DIVISIA AND FRISCH ARE FRIENDS

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L comes before P

Laspeyres price index

\[
L_P = \frac{p_1 \cdot q_0}{p_0 \cdot q_0}
\]

Paasche price index

\[
P_P = \frac{p_1 \cdot q_1}{p_0 \cdot q_1}
\]

Volume indexes

\[
L_Q = \frac{p_0 \cdot q_1}{p_0 \cdot q_0}, \quad P_Q = \frac{p_1 \cdot q_1}{p_1 \cdot q_0}
\]
Laspeyres and Paasche are Friends

1. Crossing $L$ and $P$ satisfies factor reversal:

$$L_P \cdot P_Q = \frac{p_1 \cdot q_1}{p_0 \cdot q_0} = L_Q \cdot P_P$$

2. Laspeyres and Paasche are upper and lower bounds to true cost-of-living index:

$$L_P \geq \frac{c(u_0, p_1)}{c(u_0, p_0)} \quad \text{and} \quad \frac{c(u_1, p_1)}{c(u_1, p_0)} \geq P_P$$
A Less-Well-Known Covariance Relationship

3. Difference between $P$ and $L$:

$$\frac{P_P - L_P}{L_P} = \sum_{i=1}^{n} w_{i0} \left\{ \frac{q_{i1}/q_{i0}}{L_Q} - 1 \right\} \left\{ \frac{p_{i1}/p_{i0}}{L_P} - 1 \right\},$$

where $w_{i0}$ is the budget share of good $i$ in period 0.

Examples: Next slide
Examples
4. Exactly the same for Paasche and Laspeyres volume indexes:

\[
\frac{P_Q - L_Q}{L_Q} = \sum_{i=1}^{n} w_i \left\{ \frac{q_{i1} / q_{i0}}{L_Q} - 1 \right\} \left\{ \frac{p_{i1} / p_{i0}}{L_P} - 1 \right\}
\]

This implies

\[
\frac{P_Q - L_Q}{L_Q} = \frac{P_P - L_P}{L_P}
\]
Divisia and Frisch

François Divisia
(1889 – 1964)

Ragnar Frisch
(1895 – 1973)
Divisia Indexes: Two Examples

1. **Productivity Measurement**

\[
TFP = \text{growth in output} - \text{growth in inputs}
\]

\[
= d(\log Y) - \sum_{i=1}^{n} \left( \frac{p_i q_i}{\sum_{j=1}^{n} p_j q_j} \right) d(\log q_i)
\]

where the term \( \sum_{i=1}^{n} \left( \frac{p_i q_i}{\sum_{j=1}^{n} p_j q_j} \right) d(\log q_i) \) is the Divisia index of inputs.

2. **Consumer Theory**

Budget constraint in changes: \( dM = \sum_{i=1}^{n} p_i \, dq_i + \sum_{i=1}^{n} q_i \, dp_i \), or

\[
d(\log M) = d(\log P) + d(\log Q), \text{ where}
\]

\[
d(\log P) = \sum_{i=1}^{n} w_i \, d(\log p_i), \quad d(\log Q) = \sum_{i=1}^{n} w_i \, d(\log q_i),
\]

are Divisia price and volume indexes.
Frisch Indexes

• The income elasticity of good i is

\[ \eta_i = \frac{\partial (\log q_i)}{\partial (\log M)} = \frac{\partial (p_i q_i)/\partial M}{p_i q_i/M} = \frac{Marginal \ share \ of \ i}{Budget \ share \ of \ i} = \frac{\theta_i}{w_i} \]

• Frisch uses the \( \theta_i \)'s as weights:

\[ d(\log Q') = \sum_{i=1}^{n} \theta_i d(\log q_i) \]
Luxuries and Necessities

• Frisch and Divisia:

\[ d(\log Q') = \sum_{i=1}^{n} \theta_i d(\log q_i), \quad d(\log Q) = \sum_{i=1}^{n} w_i d(\log q_i) \]

• Good a luxury when its income elasticity

\[ \eta_i = \frac{\theta_i}{w_i} > 1, \text{ or } \theta_i > w_i \]

• Thus, luxuries more heavily weighted in Frisch than Divisia; necessities less
Divisia Well-Known, Frisch Not

- Prominent surveys of index numbers tend to give prominence to Divisia
- Frisch referred to only briefly:
  - Frisch (1932), *New Methods of Measuring Marginal Utility*
- Citations to index-number work by Divisia and Frisch – next slide
Citations of Prominent Scholars

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No. of citations

Growth rate (% p.a)
Divisia and Frisch Members of Class of Friendly Indexes

• Three applications:
  1. Quality measurement
  2. Income sensitivity of MU of income
  3. Cost-of-living bias

• Usefulness of Divisia and Frisch indexes
Application I: Measuring Quality

• “[A commodity] is a queer thing, abounding in metaphysical subtleties and theological niceties”
  
  K. Marx

• Many ways to measure quality – all controversial
Easy Case 1:
Ken’s Cars (First and Current)
Easy Case 2

Rump Steak

Fillet steak more than twice expensive as rump
Difficult Case

Conventional vs. Keyhole Surgery

- Less infection risk
- Less blood loss
- Less pain
- Faster recovery
- Cost-effective
- Smaller scar
Alternative Approach to Quality

• Use the income elasticity:
  o Luxuries are of higher quality, necessities lower
  o When basket moves in the direction of luxuries, quality is said to improve
  o Quality worsens when the basket moves towards necessities

• A simple revealed preference measure of quality
Quality Index

• Contribution of good i to overall quality:
  \((\eta_i - 1)d(\log q_i)\)

• Average over n goods using budget shares as weights:
  \[\sum_{i=1}^{n} w_i (\eta_i - 1)d(\log q_i) = \sum_{i=1}^{n} w_i (\eta_i - 1)\{d(\log q_i) - d(\log Q)\}\]

• Weighted covariance between the income elasticities and growth in volumes

• **Index of quality** of the consumption basket
Back to Frisch and Divisia

- The covariance

\[ \sum_{i=1}^{n} w_i (\eta_i - 1) \{d(\log q_i) - d(\log Q)\} = d(\log Q') - d(\log Q) \]

- Quality index is excess of Frisch volume index over Divisia counterpart

- Dual price of quality is

\[ d(\log P') - d(\log P) = \sum_{i=1}^{n} \theta_i d(\log p_i) - \sum_{i=1}^{n} w_i d(\log p) \]
Quality Indexes, 37 OECD Countries, \(~25\) Recent Years

**Consumption**

- Mean = 0.41% pa
- Median = 0.39
- SD = 1.18

**Prices**

- Mean = -0.24% pa
- Median = -0.24
- SD = 0.62
Quality Demand Curves

OECD (37 countries)

\[ \frac{d \log Q'}{d \log P'} - \frac{d \log P'}{d \log P} \]

Slope = -1.31

ICP (176 countries)

\[ \frac{d \log Q'}{d \log P'} - \frac{d \log P'}{d \log P} \]

Rich Slope = -1.69

Poor Slope = -0.63
Application II: Income Sensitivity of MU of Income

- Marginal utility of income: \( \lambda = \lambda(M, p) > 0 \)
- Diminishing MU: \( \frac{\partial \lambda}{\partial M} < 0 \)
- Income elasticity of \( \lambda \): \( \frac{\partial (\log \lambda)}{\partial (\log M)} < 0 \)
- \( \phi = \left\{ \frac{\partial (\log \lambda)}{\partial (\log M)} \right\}^{-1} \) called the income flexibility
- \( \phi = \) average price elasticity of demand
- Frisch’s (1959) “universal atlas” of \( \phi \) values
- Also: Social opportunity cost of capital, project evaluation and choice under uncertainty
A Cross-Commodity Regression

Demand for good i

\[ Dq_i = \alpha_i DQ + \beta_i (Dp_i - DP) + \varepsilon_i, \quad i = 1, \ldots, n \]

Assume:

1. “Want independence” \( \beta_i = \phi \alpha_i \) (Frisch, 1959), so

\[ Dq_i = \alpha_i DQ + \phi \alpha_i (Dp_i - DP) + \varepsilon_i \]

2. Income elasticity \( \alpha_i = 1 \), so \( Dq_i - DQ = \phi (Dp_i - DP) + \varepsilon_i \)

OLS estimator of \( \phi \) is

\[ \hat{\phi} = \frac{1}{n} \sum_{i=1}^{n} \frac{(Dq_i - DQ)(Dp_i - DP)}{\frac{1}{n} \sum_{i=1}^{n} (Dp_i - DP)^2} = \frac{\text{price-quantity covariance}}{\text{price variance}} \]
A Weighted Regression

WLS:

$$\hat{\phi} = \frac{\sum_{i=1}^{n} w_i (Dq_i - DQ)(Dp_i - DP)}{\sum_{i=1}^{n} w_i (Dp_i - DP)^2}$$

$$= \frac{\text{Divisia price-quantity covariance}}{\text{Divisia price variance}} = \rho \cdot \frac{\sigma_q}{\sigma_p}$$

Subsequently

• Don’t assume income elasticities = 1
• Do assume want independence

Then, WLS estimator involves both Frisch and Divisia indexes
Heterotheticity

(Opposite of Homotheticity)

WLS estimator of $\phi$ at time $t$ is

$$\hat{\phi}_t = \frac{C_{pq t} - (D P'_t - D P_t)D Q_t}{V'_{pt}}, \quad t = 1, \ldots, T,$$

where

$$C_{pq t} = \sum_{i=1}^{n} \bar{w}_{it} (D p_{it} - D P'_t)(D q_{it} - D Q_t)$$

is a price-quantity covariance; $DP'_t - DP_t$ is price of quality; $D Q_t$ Divisia volume index; and $V'_{pt} = \sum_{i=1}^{n} \theta_i (D p_{it} - D P'_t)^2$ Frisch price variance
Income Flexibility, 37 OECD Countries

Mean = -0.41
Median = -0.39
SD = 0.82
Income Flexibility, Pairs of 176 ICP Countries

Mean = -0.52
Median = -0.48
SD = 1.68
Income Flexibility and Income (176 ICP Countries)

\[ y = 0.03x - 0.52 \]

\[ \phi_c \]

\[ (0.04) \quad (0.04) \]
Application III: The Cost-of-Living Bias

• CPI a Laspeyres index -- substitution bias

• True-cost-of-living index is

\[
\frac{C(u_0, p_1)}{C(u_0, p_0)} \approx \frac{\sum_{i=1}^{n} p_{i1} q_{i0}}{\sum_{i=1}^{n} p_{i0} q_{i0}} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{ij} Dp_{i1} Dp_{j1},
\]

where \(C(u, p)\) is the cost function; and

\[
\pi_{ij} = \left( \frac{p_{i0} p_{j0}}{\sum_{i=1}^{n} p_{i0} q_{i0}} \right) s_{ij}
\]

is a Slutsky coefficient \((\pi_{ij} = \pi_{ji})\) and \(s_{ij}\) the \((i, j)\)th substitution term
Frisch and the Bias

• Write the above as

\[
\frac{C(u_0, p_1)}{C(u_0, p_0)} - CPI \approx -B,
\]

\[
B = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{ij} Dp_{i1} Dp_{j1} > 0
\]

• Under want independence,

\[
\pi_{ij} = \begin{cases} 
\phi \theta_i (1 - \theta_i) & i = j \\
-\phi \theta_i \theta_j & i \neq j 
\end{cases}
\]

and the bias simplifies to

\[
B = -\frac{1}{2} \phi V_p', \quad \text{where} \quad V_p' = \sum_{i=1}^{n} \theta_i (Dp_i - DP')^2
\]

is the Frisch price variance and \( \phi \) is income flexibility
An Elegantly Simple Result

- \( CPI\ bias = -\frac{1}{2} \times income\ flexibility \times price\ dispersion \)

- When income flexibility \( \approx -\frac{1}{2} \),

\[
CPI\ bias \approx \frac{price\ dispersion}{4}
\]

Here,

\[ price\ dispersion = Frisch\ price\ variance \]

- More relative price changes, larger the substitution effects and larger the CPI bias
Harberger Triangle

\[
\frac{C(u_0, p_1)}{C(u_0, p_0)} - CPI \approx \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{ij} Dp_{i1} Dp_{j1}
\]

• Change the sign of QF:

\[
-\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{ij} Dp_{i1} Dp_{j1} > 0
\]

• This Harberger’s (1964) generalised triangle measure of welfare cost of distortions

• Thus,

\[
\text{Welfare cost} \approx CPI - TCL \approx \frac{\text{Frisch price variance}}{4}
\]
Divisia and Frisch Partnership

• Divisia and Frisch indexes both useful price and volume indexes

• Also useful in tandem:
  o Quality measurement
  o Short-cut estimates of income flexibility
  o CPI bias
  o Harberger triangle

• Divisia and Frisch “friendly”, just like Laspeyres and Paasche
REFERENCES


REFERENCES (cont’d)


REFERENCES (cont’d)


Additional Material
Selected Index-Numbers
Publications and Citations

A. Balk


Citations
70
245
95
54
56
229
220
B. Diewert


Selected Index-Numbers
Publications and Citations (cont’d)

C. Divisia

D. Fisher

E. Frisch

F. Samuelson and Swamy

Note: These items are used to construct the figure giving citations. They represent the most-cited works, on a lifetime basis, on index numbers by the respective author. Citations are from a January 2018 search using Google Scholar. Diewert’s items are restricted to those with a minimum of 200 lifetime number citations.