# DIVISIA AND FRISCH ARE FRIENDS

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# L comes before P

*Laspeyres price index*

$$= \frac{Yesterday's\ basket\ at\ today's\ prices}{Yesterday's\ basket\ at\ yesterday's\ prices}$$

$$L_P = \frac{\boldsymbol{p}_1 \cdot \boldsymbol{q}_0}{\boldsymbol{p}_0 \cdot \boldsymbol{q}_0}$$

*Paasche price index*

$$= \frac{Today's\ basket\ at\ today's\ prices}{Today's\ basket\ at\ yesterday's\ prices} = P_P = \frac{\boldsymbol{p}_1 \cdot \boldsymbol{q}_1}{\boldsymbol{p}_0 \cdot \boldsymbol{q}_1}$$

*Volume indexes*

$$L_Q = \frac{\boldsymbol{p}_0 \cdot \boldsymbol{q}_1}{\boldsymbol{p}_0 \cdot \boldsymbol{q}_0}, \qquad P_Q = \frac{\boldsymbol{p}_1 \cdot \boldsymbol{q}_1}{\boldsymbol{p}_1 \cdot \boldsymbol{q}_0}$$

# Laspeyres and Paasche are Friends

1. Crossing *L* and *P* satisfies factor reversal:

$$L_P \cdot P_Q = \frac{\boldsymbol{p}_1 \cdot \boldsymbol{q}_1}{\boldsymbol{p}_0 \cdot \boldsymbol{q}_0} = L_Q \cdot P_P$$

2. Laspeyres and Paasche are upper and lower bounds to true cost-of-living index:

$$L_P \geq \frac{C(u_0, \boldsymbol{p}_1)}{C(u_0, \boldsymbol{p}_0)} \text{ and } \frac{C(u_1, \boldsymbol{p}_1)}{C(u_1, \boldsymbol{p}_0)} \geq P_P$$

# A Less-Well-Known Covariance Relationship

3. Difference between *P* and *L*:

$$\frac{P_P - L_P}{L_P} = \sum_{i=1}^{n} w_{i0} \left\{ \frac{q_{i1}/q_{i0}}{L_Q} - 1 \right\} \left\{ \frac{p_{i1}/p_{i0}}{L_P} - 1 \right\},$$

where $w_{i0}$ is the budget share of good i in period 0

Examples: Next slide

# Examples

# Same for Difference between Volume Indexes

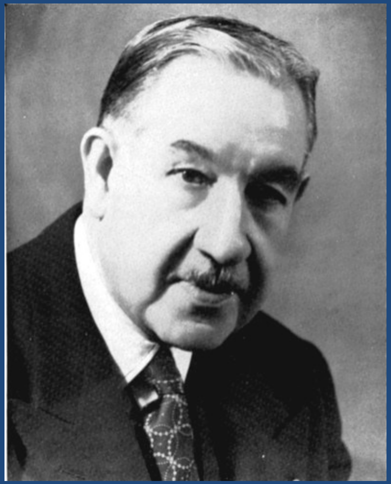
4. Exactly the same for Paasche and Laspeyres volume indexes:

$$\frac{P_Q - L_Q}{L_Q} = \sum_{i=1}^{n} w_{i0} \left\{ \frac{q_{i1}/q_{i0}}{L_Q} - 1 \right\} \left\{ \frac{p_{i1}/p_{io}}{L_P} - 1 \right\}$$

This implies

$$\frac{P_Q - L_Q}{L_Q} = \frac{P_P - L_P}{L_P}$$

# Divisia and Frisch



François Divisia
(1889 – 1964)

Ragnar Frisch
(1895 – 1973)

# Divisia Indexes: Two Examples

1. Productivity Measurement

$$TFP = growth\ in\ output\ - growth\ in\ inputs$$

$$= d(\log Y) - \underbrace{\sum_{i=1}^{n} \left( \frac{p_i q_i}{\sum_{j=1}^{n} p_j q_j} \right) d(\log q_i)}_{\text{Divisia index of inputs}}$$

2. Consumer Theory

Budget constraint in changes: $dM = \sum_{i=1}^{n} p_i\, dq_i + \sum_{i=1}^{n} q_i\, dp_i$, or

$$d(\log M) = d(\log P) + d(\log Q) \text{, where}$$

$$d(\log P) = \sum_{i=1}^{n} w_i d(\log p_i), \qquad d(\log Q) = \sum_{i=1}^{n} w_i d(\log q_i),$$

are Divisia price and volume indexes

# Frisch Indexes

- The income elasticity of good i is

$$\eta_i = \frac{\partial(\log q_i)}{\partial(\log M)} = \frac{\partial(p_i q_i)/\partial M}{p_i q_i/M}$$

$$= \frac{Marginal\ share\ of\ i}{Budget\ share\ of\ i} = \frac{\theta_i}{w_i}$$

- Frisch uses the $\theta_i' s$ as weights:

$$d(\log Q') = \sum_{i=1}^{n} \theta_i d(\log q_i)$$

# Luxuries and Necessities

- Frisch and Divisia:

$$d(\log Q') = \sum_{i=1}^{n} \theta_i d(\log q_i), \;\; d(\log Q) = \sum_{i=1}^{n} w_i d(\log q_i)$$

- Good a luxury when its income elasticity

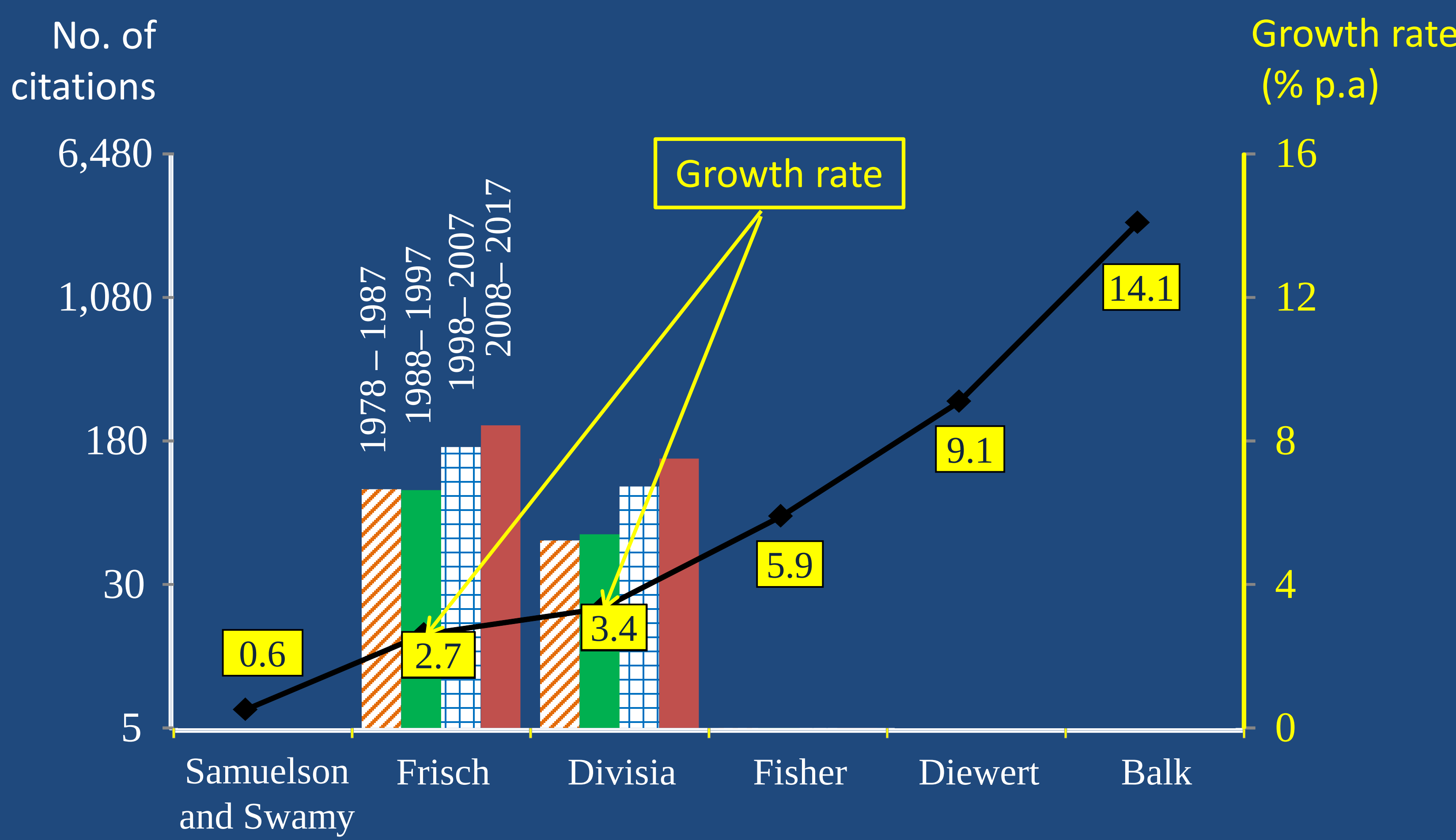
$$\eta_i = \frac{\theta_i}{w_i} > 1, \text{ or } \theta_i > w_i$$

- Thus, luxuries more heavily weighted in Frisch than Divisia; necessities less

# Divisia Well-Known, Frisch Not

- Prominent surveys of index numbers tend to give prominence to Divisia
- Frisch referred to only briefly:
    - Frisch (1932), New Methods of Measuring Marginal Utility
    - Frisch (1936), "Annual Survey of General Economic Theory: The Problem of Index Numbers." Econometrica
- Citations to index-number work by Divisia and Frisch – next slide

# Citations of Prominent Scholars



# Divisia and Frisch Members of Class of Friendly Indexes

- Three applications:
  1. Quality measurement
  2. Income sensitivity of MU of income
  3. Cost-of-living bias
- Usefulness of Divisia and Frisch indexes

# Application I: Measuring Quality

- “[A commodity] is a queer thing, abounding in metaphysical subtleties and theological niceties”

  K. Marx

- Many ways to measure quality – all controversial

# Easy Case 1:
# Ken’s Cars (First and Current)

# Easy Case 2

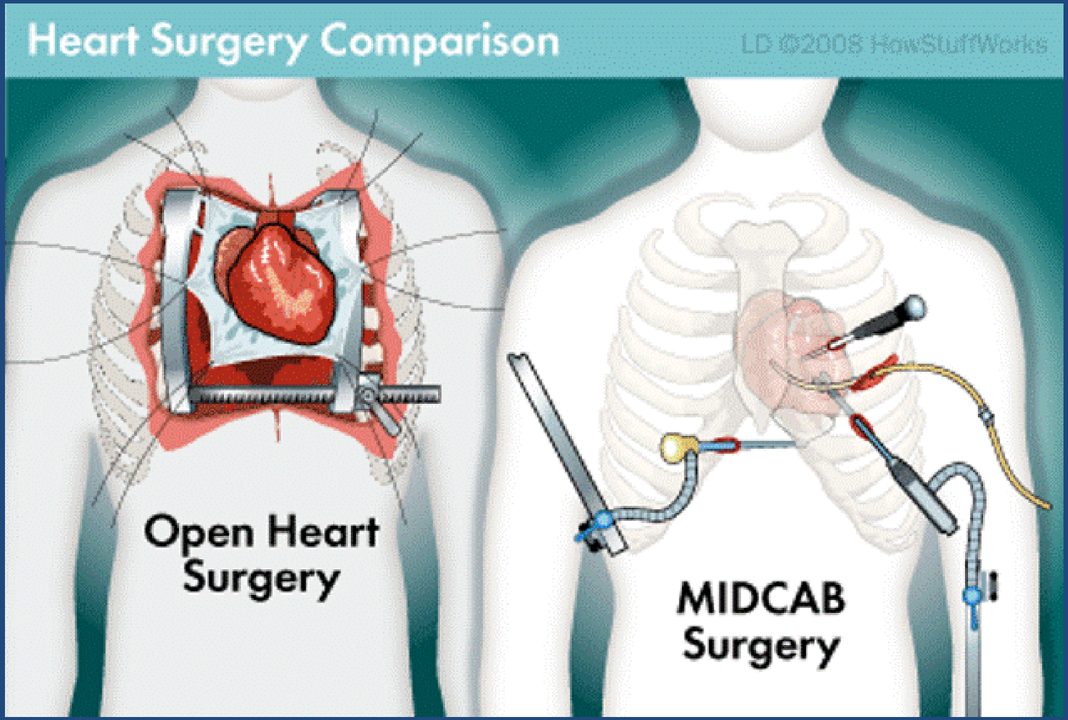

Versus


**Rump Steak**

**Fillet Steak**

Fillet steak more than twice expensive as rump

# Difficult Case




## Conventional vs. Keyhole Surgery

- Less infection risk
- Less blood loss
- Less pain
- Faster recovery
- Cost-effective
- Smaller scar

# Alternative Approach to Quality

- Use the income elasticity:
  - Luxuries are of higher quality, necessities lower
  - When basket moves in the direction of luxuries, quality is said to improve
  - Quality worsens when the basket moves towards necessities
- A simple revealed preference measure of quality

# Quality Index

- Contribution of good i to overall quality:

$$(\eta_i - 1)d(\log q_i)$$

- Average over n goods using budget shares as weights:

$$\sum_{i=1}^{n} w_i(\eta_i - 1)d(\log q_i) = \sum_{i=1}^{n} w_i(\eta_i - 1)\{d(\log q_i) - d(\log Q)\}$$

- Weighted covariance between the income elasticities and growth in volumes
- Index of quality of the consumption basket

# Back to Frisch and Divisia

- The covariance

$$\sum_{i=1}^{n} w_i(\eta_i - 1)\{d(\log q_i) - d(\log Q)\}$$

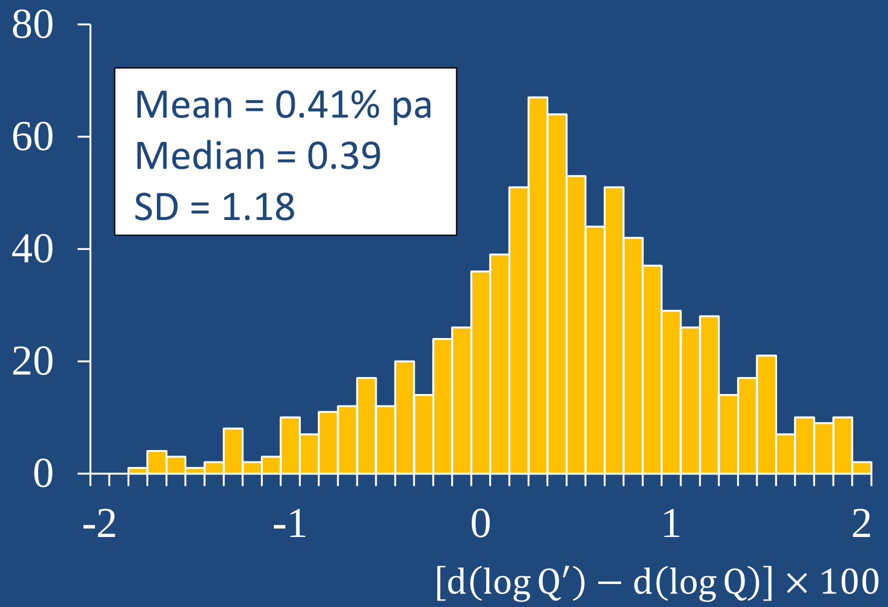
$$= d(\log Q') - d(\log Q)$$

- Quality index is excess of Frisch volume index over Divisia counterpart
- Dual price of quality is

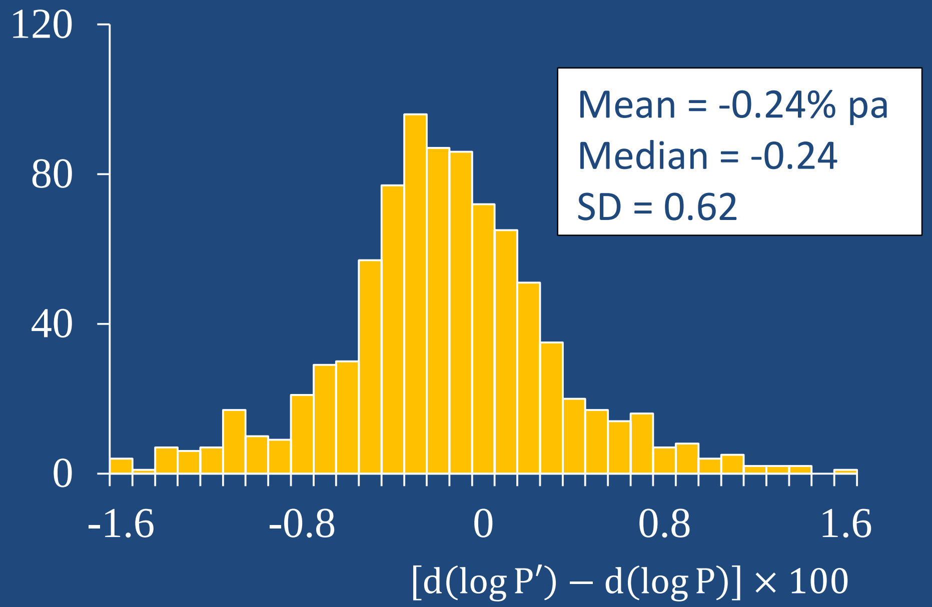
$$d(\log P') - d(\log P) = \sum_{i=1}^{n} \theta_i d(\log p_i) - \sum_{i=1}^{n} w_i d(\log p)$$

# Quality Indexes, 37 OECD Countries, ~25 Recent Years

Consumption

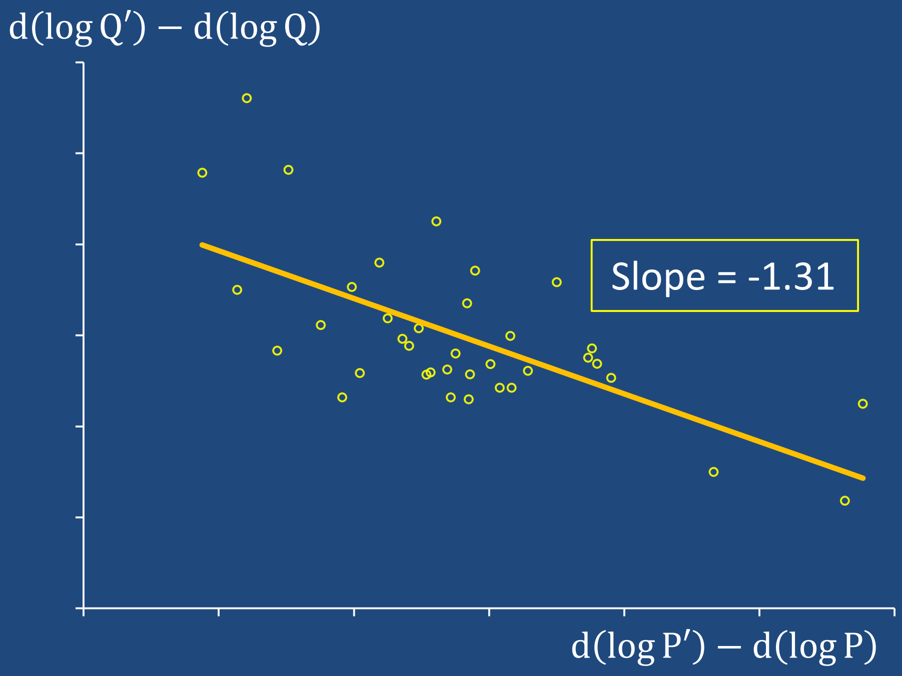



Prices

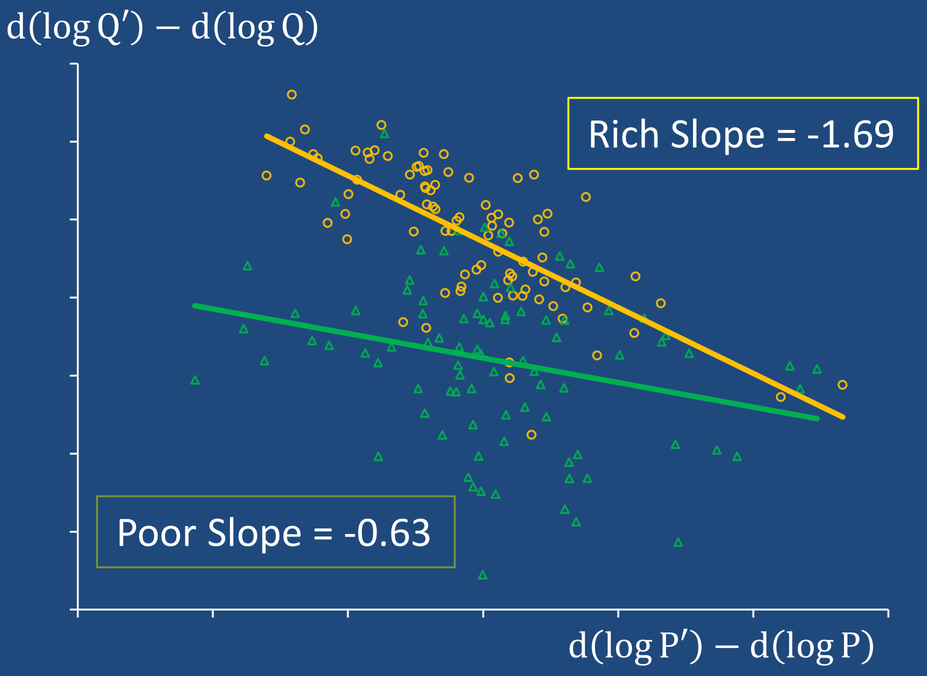


# Quality Demand Curves

OECD (37 countries)




ICP (176 countries)



# Application II: Income Sensitivity of MU of Income

- Marginal utility of income: $\lambda = \lambda(M, \boldsymbol{p}) > 0$
- Diminishing MU: $\frac{\partial \lambda}{\partial M} < 0$
- Income elasticity of $\lambda$: $\frac{\partial(\log \lambda)}{\partial(\log M)} < 0$
- $\phi = \left\{\frac{\partial(\log \lambda)}{\partial(\log M)}\right\}^{-1}$ called the income flexibility
- $\phi$ = average price elasticity of demand
- Frisch's (1959) "universal atlas" of $\phi$ values
- Also: Social opportunity cost of capital, project evaluation and choice under uncertainty

# A Cross-Commodity Regression

Demand for good i

$$Dq_i = \alpha_i DQ + \beta_i(Dp_i - DP) + \varepsilon_i\ , \qquad i = 1, \cdots, n$$

Assume:

1. "Want independence" $\beta_i = \phi\alpha_i$ (Frisch, 1959), so

$$Dq_i = \alpha_i DQ + \phi\alpha_i(Dp_i - DP) + \varepsilon_i$$

2. Income elasticity $\alpha_i = 1$, so $Dq_i - DQ = \phi(Dp_i - DP) + \varepsilon_i$

OLS estimator of $\phi$ is

$$\hat{\phi} = \frac{\frac{1}{n}\sum_{i=1}^{n}(Dq_i - DQ)(Dp_i - DP)}{\frac{1}{n}\sum_{i=1}^{n}(Dp_i - DP)^2} = \frac{price\text{–}quantity\ covariance}{price\ variance}$$

# A Weighted Regression

WLS:

$$\hat{\phi} = \frac{\sum_{i=1}^{n} w_i(Dq_i - DQ)(Dp_i - DP)}{\sum_{i=1}^{n} w_i(Dp_i - DP)^2}$$

$$= \frac{\textit{Divisia price–quantity covariance}}{\textit{Divisia price variance}} = \rho \cdot \frac{\sigma_q}{\sigma_p}$$

Subsequently

- Don't assume income elasticities = 1
- Do assume want independence

Then, WLS estimator involves both Frisch and Divisia indexes

# Heterotheticity

## (Opposite of Homotheticity)

WLS estimator of $\phi$ at time t is

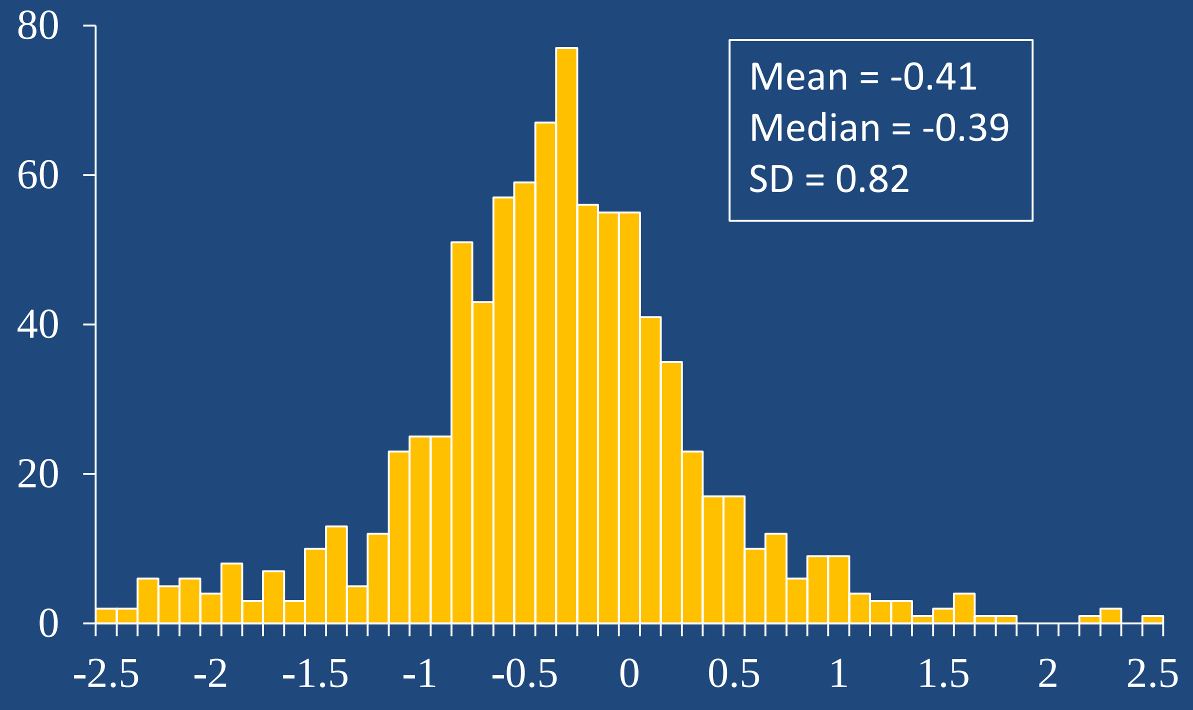
$$\hat{\phi}_t = \frac{C_{pqt} - (DP'_t - DP_t)DQ_t}{\mathrm{V}'_{pt}}, \qquad t = 1, \cdots, T,$$

where

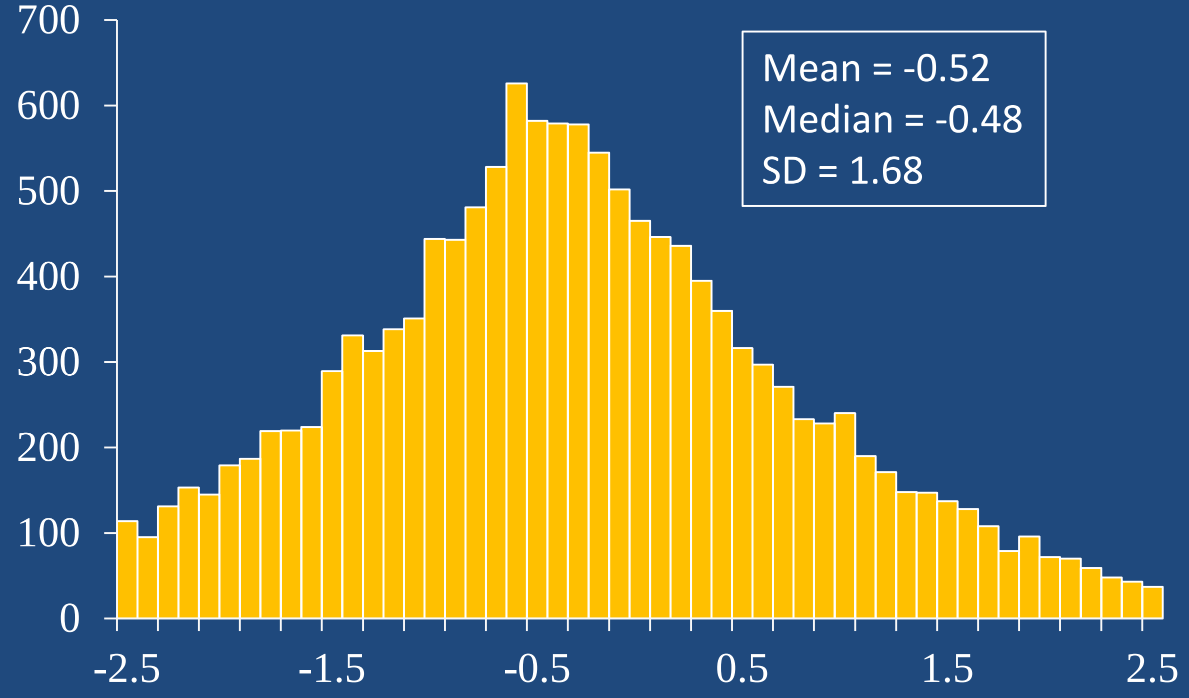
$$C_{pqt} = \sum_{i=1}^{n} \bar{w}_{it}(Dp_{it} - DP'_t)\,(Dq_{it} - DQ_t)$$

is a price-quantity covariance; $DP'_t - DP_t$ is price of quality; $DQ_t$ Divisia volume index; and $\mathrm{V}'_{pt} = \sum_{i=1}^{n} \theta_i (Dp_{it} - DP'_t)^2$ Frisch price variance

# Income Flexibility, 37 OECD Countries

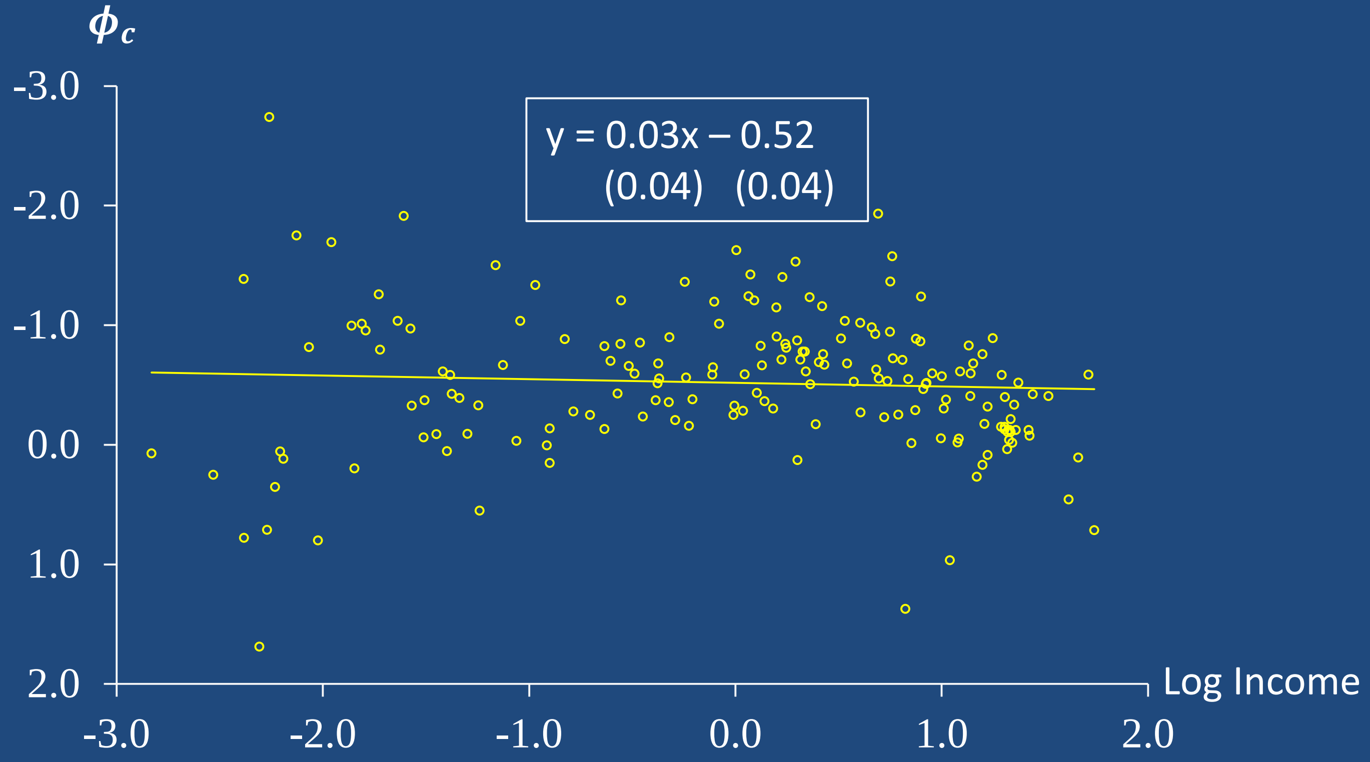


# Income Flexibility, Pairs of 176 ICP Countries



# Income Flexibility and Income

(176 ICP Countries)



# Application III: The Cost-of-Living Bias

- CPI a Laspeyres index -- substitution bias
- True-cost-of-living index is

$$\frac{C(u_0, \boldsymbol{p}_1)}{C(u_0, \boldsymbol{p}_0)} \approx \frac{\sum_{i=1}^{n} p_{i1} q_{i0}}{\sum_{i=1}^{n} p_{i0} q_{i0}} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{ij} Dp_{i1} Dp_{j1} ,$$

where $C(u, \boldsymbol{p})$ is the cost function; and

$$\pi_{ij} = \left( \frac{p_{i0} p_{j0}}{\sum_{i=1}^{n} p_{i0} q_{i0}} \right) s_{ij}$$

is a Slutsky coefficient $(\pi_{ij} = \pi_{ji})$ and $s_{ij}$ the $(i, j\ )^{th}$ substitution term

# Frisch and the Bias

- Write the above as

$$\frac{C(u_0, \boldsymbol{p}_1)}{C(u_0, \boldsymbol{p}_0)} - CPI \approx -B, \qquad B = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \pi_{ij} Dp_{i1} Dp_{j1} > 0$$

- Under want independence,

$$\pi_{ij} = \begin{cases} \phi\theta_i(1-\theta_i) & i = j \\ -\phi\theta_i\theta_j & i \neq j \end{cases}$$

and the bias simplifies to

$$B = -\frac{1}{2}\phi V_p', \qquad \text{where} \quad V_p' = \sum_{i=1}^{n} \theta_i \left(Dp_i - DP'\right)^2$$

is the Frisch price variance and $\phi$ is income flexibility

# An Elegantly Simple Result

- $CPI\ bias = -\frac{1}{2} \times income\ flexibility \times price\ dispersion$

- When income flexibility $\approx -\frac{1}{2}$,

$$\boxed{CPI\ bias \approx \frac{price\ dispersion}{4}}$$

Here,

$$price\ dispersion = Frisch\ price\ variance$$

- More relative price changes, larger the substitution effects and larger the CPI bias

# Harberger Triangle

$$\frac{C(u_0, \boldsymbol{p}_1)}{C(u_0, \boldsymbol{p}_0)} - CPI \approx \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_{ij}Dp_{i1}Dp_{j1}$$

- Change the sign of QF:

$$-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_{ij}Dp_{i1}Dp_{j1} > 0$$

- This Harberger's (1964) generalised triangle measure of welfare cost of distortions
- Thus,

$$\boxed{Welfare\ cost\ \approx CPI - TCL \approx \frac{Frisch\ price\ variance}{4}}$$

# Divisia and Frisch Partnership

- Divisia and Frisch indexes both useful price and volume indexes
- Also useful in tandem:
    - Quality measurement
    - Short-cut estimates of income flexibility
    - CPI bias
    - Harberger triangle
- Divisia and Frisch “friendly”, just like Laspeyres and Paasche

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Note: These items are used to construct the figure giving citations. They represent the most-cited works, on a lifetime basis, on index numbers by the respective author. Citations are from a January 2018 search using Google Scholar. Diewert's items are restricted to those with a minimum of 200 lifetime number citations.