Chapter 9 Seasonal Products

CONSUMER PRICE INDEX THEORY

Chapter Titles

CHAPTER 9: SEASONAL PRODUCTS[1]

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Table of Contents

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1. The Problem of Seasonal Products

The existence of seasonal products or commodities poses some significant challenges for price statisticians. *Seasonal commodities* are commodities which are either: (a) not available in the marketplace during certain seasons of the year or (b) are available throughout the year but there are regular fluctuations in prices or quantities that are synchronized with the season or the time of the year.[2] A commodity that satisfies (a) is termed a *strongly seasonal commodity* whereas a commodity which satisfies (b) is a *weakly seasonal commodity*. It is strongly seasonal commodities that create the biggest problems for price statisticians in the context of producing a monthly or quarterly Consumer Price Index because if a commodity price is available in only one of the two months (or quarters) being compared, then obviously it is not possible to calculate a relative price for the commodity and traditional bilateral index number theory breaks down. In other words, if a commodity is present in one month but not the next, how can the month to month amount of price change for that commodity be computed?[3] There is no easy solution to this lack of comparability problem. This chapter will present various attempts at finding solutions to this problem.

There are two main sources of seasonal fluctuations in prices and quantities: (a) climate and (b) custom.[4] In the first category, fluctuations in temperature, precipitation and hours of daylight cause fluctuations in the demand or supply for many commodities; e.g., think of summer versus winter clothing, the demand for light and heat, vacations, etc. With respect to custom and convention as a cause of seasonal fluctuations, consider the following quotation:

"Conventional seasons have many origins—ancient religious observances, folk customs, fashions, business practices, statute law… Many of the conventional seasons have considerable effects on economic behaviour. We can count on active retail buying before Christmas, on the Thanksgiving demand for turkeys, on the first of July demand for fireworks, on the preparations for June weddings, on heavy dividend and interest payments at the beginning of each quarter, on an increase in bankruptcies in January, and so on." Wesley C. Mitchell (1927; 237).

Examples of important seasonal commodities are: many food items; alcoholic beverages; many clothing and footwear items; water; heating oil; electricity; flowers and garden supplies; vehicle purchases; vehicle operation; many entertainment and recreation expenditures; books, insurance expenditures; wedding expenditures; recreational equipment; air travel and tourism expenditures. For a "typical" country, seasonal expenditures will often amount to one fifth to one third of all consumer expenditures.[5]

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[2] This classification of seasonal commodities corresponds to Balk's narrow and wide sense seasonal commodities; see Balk (1980a; 7) (1980b; 110) (1980c; 68). Diewert (1998; 457) used the terms type 1 and type 2 seasonality.

[3] Zarnowitz (1961; 238) was perhaps the first to note the importance of this problem: "But the main problem introduced by the seasonal change is precisely that the market basket is different in the consecutive months (seasons), not only in weights but presumably often also in its very composition by commodities. This is a general and complex problem which will have to be dealt with separately at later stages of our analysis."

[4] This classification dates back to Mitchell (1927; 236) at least: "Two types of seasons produce annually recurring variations in economic activity--those which are due to climates and those which are due to conventions."

[5] Alterman, Diewert and Feenstra (1999; 151) found that over the 40 months between September 1993 and December 1996, somewhere between 23 and 40 percent of U.S. imports and exports exhibited seasonal variations in *quantities* whereas only about 5 percent of U.S. export and import *prices* exhibited seasonal fluctuations.

In the context of producing a monthly or quarterly Consumer Price Index, it must be recognized that there is no completely satisfactory way for dealing with strongly seasonal commodities. If a commodity is present in one month but missing from the market place in the next month, then many of the index number theories that were considered in earlier chapters cannot be applied because these theories assumed that the dimensionality of the commodity space was constant for the two periods being compared. However, if seasonal commodities are present in the market for certain months of the year on a regular basis, then traditional index number theory can be applied in order to construct *year over year indexes for the same month*. This approach is discussed in sections 2 and 3 below. In the initial sections of this chapter, it will be assumed that price and quantity information for the seasonal commodities is available. The various indexes which are considered in this chapter will be illustrated using actual data on fresh fruit consumption for Israel for the 6 years 2012-2017. The underlying data are listed in the Appendix along with tables using these data that list the various indexes that are considered in the main text.

The methods that are suggested in sections 2-6 of this chapter to deal with seasonal commodities assume that the statistical agency is able to collect *price* and *expenditure* information on these seasonal commodities by month.[6] In sections 7 and 8, the construction of month to month indexes using *only price information* will be considered.

The indexes discussed in the various sections of this chapter are different depending on the following differences that characterize the method used to deal with the seasonality problem and the availability of data:

- Price and quantity (or expenditure) data are available versus only price information is available.
- Carry forward prices are used as imputations for missing prices versus methods that do not use imputations.
- A year over year index for the same month is constructed versus a month to month index is constructed. Annual indexes that measure all prices in one year relative to another year provide another source of difference.
- A traditional fixed base or chained bilateral Laspeyres, Paasche, Fisher or Törnqvist index are constructed versus the use of a multilateral index.
- The index uses monthly weights or it uses annual weights.

With the above five main sources of differences in index concept in mind, an overview of the various sections in this chapter follows.

Sections 2-7 deal with methods that make use of monthly *price* and *quantity* information. Section 2 constructs traditional fixed base and chained *year over year monthly indexes* using year over year *carry forward prices* for any missing prices. Section 3 *constructs year over year monthly indexes* using fixed base or chained or multilateral indexes *without using imputations* for missing prices and quantities. Sections 4 and 5 consider the production of *annual indexes*. These annual indexes treat each monthly commodity as a separate commodity in a yearly index. The section 4

[6] Hardly any statistical agencies have monthly expenditure surveys and so many of the methods suggested in this chapter are not feasible at present. However, an increasing number of agencies are collecting weekly scanner data from retailers which have detailed price and quantity information on sales by individual product, including seasonal products. In addition, in the future, it may become possible to collect electronic data on consumer products directly from households. Thus in the future, it will be possible to implement the methods suggested in this chapter for at least parts of a country's CPI.

indexes use carry forward prices for missing prices while the section 5 indexes do not use any imputed prices. It turns out that some of the Laspeyres and Paasche annual indexes that use carry forward prices can be related to the year over year Laspeyres and Paasche monthly indexes studied in sections 2 and 3.

Section 6 constructs traditional *month to month indexes* using carry forward prices for the missing prices. Section 7 constructs month to month fixed base and chained Laspeyres, Paasche and Fisher indexes as well as some multilateral indexes (with no imputations for missing prices).

Sections 8 and 9 construct indexes *using only information on prices*. Section 8 uses carry forward prices for the missing prices while section 9 uses multilateral methods with no imputations for any missing prices. A new multilateral method of linking price observations based on *relative price similarity* is suggested in section 9.

Section 10 assumes that some expenditure or quantity information is available in addition to price information. The additional expenditure information that is assumed available is *annual expenditure information by product for a base year*. With this extra information (and the use of carry forward prices for any missing prices), a Lowe (1823) or Young (1812) index can be calculated and compared to some of the alternative indexes that were calculated in earlier sections.

Section 11 returns to the problems associated with forming annual indexes. The annual indexes to be studied in sections 4 and 5 are annual indexes for calendar years. In section 11, these annual indexes are generalized to form Rolling Year annual indexes; i.e., the prices of 12 consecutive months are compared with the prices of a base period run of 12 consecutive months and the price comparisons are such that the January prices in the current rolling year are compared with the January prices in the base year; the February prices in the currrent rolling year are compared with the February prices in the base year and so on. It turns out that these Rolling Year indexes are related to measures of trend inflation.

Section 12 concludes by summarizing the more important results in the light of the calculations using the Israeli data set.

Before proceeding to the technical definitions of the various indexes, it is useful to discuss the notation that will be used and the interpretation of the variables. The algebra below assumes that the statistical agency has information on the monthly prices and quantities for the N commodities that enter the scope of the index. However, not all commodities will be present in each month. Denote the set of commodities n which are present in the marketplace during month m of year y as S(y,m). Data on prices and quantities are available for Y years and say M = 12 months.[7] Denote the price of commodity n in month m of year y as $p_{y,m,n}$ the corresponding quantity as $q_{y,m,n}$ and the corresponding expenditure share as:

(1) $s_{y,m,n} \equiv p_{y,m,n}q_{y,m,n}/\sum_{k\in S(y,m)} p_{y,m,k}q_{y,m,k}$; $\quad y = 1,\ldots,Y;\ m = 1,2,\ldots,M\ ;\ n\in S(y,m).$[8]

[7] It is possible to construct "monthly" indexes that consist of 13 "months" that consist of 4 consecutive weeks. Thus when we define various indexes, we will generally assume that there are data for M "months" in the year. This also allows M to equal 4 for cases where quarterly price indexes are constructed. However, for our empirical example, M = 12.

[8] The summation $\sum_{k\in S(y,m)} p_{y,m,k}q_{y,m,k}$ means that we sum expenditures in month m of year y over products k that are actually present in month m of year y; i.e., strongly seasonal products that are not present in month m of year y are excluded in this sum.

It is assumed that $q_{y,m,n}$ is the total quantity of product n sold to households in scope for the index in month m of year y and $p_{y,m,n}$ is the corresponding monthly unit value price. In the following four sections, various index number formulae will be defined using the above notation. However, the resulting indexes could refer to several situations:

- N is the total number of separate items that are to be distinguished in the overall consumer price index; i.e., the underlying assumption here is that we have complete price and quantity information on the universe of expenditures for the reference population.
- N refers to the number of items in one particular *stratum* of the overall consumer price index. Standard index number theory is also applicable in this situation.
- The various methodologies to deal with seasonal commodities could be applied at higher levels of aggregation. Data on expenditures by category could be available along with elementary price indexes for the categories in scope. Implicit quantities (or volumes) by category could be constructed by deflating the expenditure categories by the respective elementary price indexes. These deflated expenditures are treated as the quantities $q_{y,m,n}$ and the corresponding elementary price indexes $p_{y,m,n}$ are treated as the corresponding prices.

Obviously, application of the first interpretation of the indexes is unrealistic; the statistical agency will typically not have access to true microeconomic data at the finest level of aggregation. However, application of the second interpretation of the indexes is quite possible; the existence of scanner data sets has led to the possibility of computing say true Fisher indexes for some strata of the CPI.[9]

2. Year over Year Monthly Indexes using Carry Forward Prices

For over a century,[10] it has been recognized that making year over year comparisons[11] of prices in the same month provides the simplest method for making comparisons that are (mostly) free from the contaminating effects of seasonal fluctuations. For example, the economist Flux and the statistician Yule endorsed the idea of making year over year comparisons to minimize the effects of seasonal fluctuations:

> "Each month the average price change compared with the corresponding month of the previous year is to be computed. … The determination of the proper seasonal variations of weights, especially in view of the liability of seasons to vary from year to year, is a task from which, I imagine, most of us would be tempted to recoil." A. W. Flux (1921; 184-185).

> "My own inclination would be to form the index number for any month by taking ratios to the corresponding month of the year being used for reference, the year before presumably, as this would avoid any difficulties with seasonal commodities. I should then form the annual average by the geometric mean of the monthly figures." G. Udny Yule (1921; 199).

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[9] See Ivancic, Diewert and Fox (2011), de Haan and van der Grient (2011) and the Australian Bureau of Statistics (2016) for applications of this type.

[10] "In the daily market reports, and other statistical publications, we continually find comparisons between numbers referring to the week, month, or other parts of the year, and those for the corresponding parts of a previous year. The comparison is given in this way in order to avoid any variation due to the time of the year. And it is obvious to everyone that this precaution is necessary. Every branch of industry and commerce must be affected more or less by the revolution of the seasons, and we must allow for what is due to this cause before we can learn what is due to other causes." W. Stanley Jevons (1884;3).

[11] In the seasonal price index literature, this type of index corresponds to Bean and Stine's (1924; 31) Type D index.

Zarnowitz also endorsed the use of year over year monthly indexes:

"There is of course no difficulty in measuring the average price change between the same months of successive years, if a month is our unit 'season', and if a constant seasonal market basket can be used, for traditional methods of price index construction can be applied in such comparisons." Victor Zarnowitz (1961; 266).

However, using year over year monthly indexes does not completely solve the seasonality problem. Diewert, Finkel and Artsev found that strongly seasonal fresh fruits in Israel did not always appear in the same months:[12]

"Seasonal fluctuations are not completely synchronized with the calendar months for products with strong seasonality. Thus a product may appear/disappear a month before/later than in the previous year." W. Erwin Diewert, Yoel Finkel and Yevgeny Artsev (2011; 63).

In the present section, we will deal with the possibility that the strongly seasonal products do not always appear in the same month of each year by using *carry forward prices* from the previous year (for the same month)[13] for any missing prices. The corresponding missing quantities are set equal to 0. With these conventions, the set of available products for month m in year y, S(y,m), is defined to include any temporarily missing products so that for any month m, the set of available products for month m in year y will always be the same. Thus the set of "available" products for month m in year y, S(y,m), will be constant over the years y.[14] Thus we can denote the common set of "available" products for month m in any year y as S(m). With this new notation that accommodates the carry forward prices for missing products in a given month, the *fixed base Laspeyres, Paasche, Fisher* and *Törnqvist-Theil indexes* [15]for month m in year y are defined as follows:

(2) $P_{LFB}^{y,m} \equiv \sum_{n\in S(m)} (p_{y,m,n}/p_{1,m,n})s_{1,m,n}$; $\quad m = 1,...,M; y = 1,...,Y;$

(3) $P_{PFB}^{y,m} \equiv [\sum_{n\in S(m)} (p_{y,m,n}/p_{1,m,n})^{-1}s_{y,m,n}]^{-1}$; $\quad m = 1,...,M; y = 1,...,Y;$

(4) $P_{FFB}^{y,m} \equiv [P_{LFB}^{y,m}\, P_{PFB}^{y,m}]^{1/2}$; $\quad m = 1,...,M; y = 1,...,Y;$

(5) $P_{TFB}^{y,m} \equiv \exp[\sum_{n\in S(m)} (\frac{1}{2})(s_{1,m,n}+ s_{y,m,n})\ln(p_{y,m,n}/p_{1,m,n})]$; $\quad m = 1,...,M; y = 1,...,Y.$

The expenditure shares, $s_{y,m,n}$, which appear in definitions (2)-(5) are defined above by definitions (1).

The chained versions of the above four indexes are defined in two stages. For the first stage, define the *chain link* for each of the above indexes going from month m in year y−1 to month m in year y as follows:

[12] A similar lack of matching problem can occur if national holidays do not always appear in the same month of the year.

[13] If the missing product is missing in the previous year (for the same month), go backwards in time to the last year (for the same month) when the product was present. If the product was not present (for the same month) in any previous year, go to the year when the product first appears in the month under consideration and use this price as a carry backward price for the years that the product was missing.

[14] Thus for this section where we use year over year carry forward (or backward) prices for any strongly seasonal products that happen to be missing in one or more years, the set of "available" products in month m for any year in our sample is the set of products that appeared in at least one month m over all month m's in the sample of years.

[15] See Laspeyres (1871), Paasche (1874), Fisher (1922), Törnqvist (1936), Törnqvist and Törnqvist (1937) and Theil (1967).

(6) $P_{LLINK}{}^{y,m} \equiv \sum_{n \in S(m)} (p_{y,m,n}/p_{y-1,m,n})s_{y-1,m,n}$; $m = 1,...,M; y = 2,...,Y;$
(7) $P_{PLINK}{}^{y,m} \equiv [\sum_{n \in S(m)} (p_{y,m,n}/p_{y-1,m,n})^{-1}s_{y,m,n}]^{-1}$; $m = 1,...,M; y = 2,...,Y;$
(8) $P_{FLINK}{}^{y,m} \equiv [P_{LLINK}{}^{y,m}\, P_{PLINK}{}^{y,m}]^{1/2}$; $m = 1,...,M; y = 2,...,Y;$
(9) $P_{TLINK}{}^{y,m} \equiv \exp[\sum_{n \in S(m)} (½)(s_{y-1,m,n} + s_{y,m,n})\ln(p_{y,m,n}/p_{y-1,m,n})]$; $m = 1,...,M; y = 2,...,Y.$

Define the *chained Laspeyres, Paasche, Fisher* and *Törnqvist-Theil indexes* for month m in year 1 as unity:

(10) $P_{LCH}{}^{1,m} \equiv 1$; $P_{PCH}{}^{1,m} \equiv 1$; $P_{FCH}{}^{1,m} \equiv 1$; $P_{TCH}{}^{1,m} \equiv 1$; $m = 1,....,M.$

For years following year 1, the above indexes are defined by cumulating the corresponding chain links; i.e., we have the following definitions:

(11) $P_{LCH}{}^{y,m} \equiv P_{LCH}{}^{y-1,m}\, P_{LLINK}{}^{y,m}$; $m = 1,...,M; y = 2,...,Y;$
(12) $P_{PCH}{}^{y,m} \equiv P_{PCH}{}^{y-1,m}\, P_{PLINK}{}^{y,m}$; $m = 1,...,M; y = 2,...,Y;$
(13) $P_{FCH}{}^{y,m} \equiv P_{FCH}{}^{y-1,m}\, P_{FLINK}{}^{y,m}$; $m = 1,...,M; y = 2,...,Y;$
(14) $P_{TCH}{}^{y,m} \equiv P_{TCH}{}^{y-1,m}\, P_{TLINK}{}^{y,m}$; $m = 1,...,M; y = 2,...,Y.$

For each month m, there are eight commonly used indexes to choose from. From the viewpoint of the economic approach to index number theory, the two Laspeyres indexes are subject to some upward substitution bias relative to a cost of living index while the two Paasche indexes are subject to some downward substitution bias. If there are smooth trends in prices and quantities, these substitution biases will be lower in magnitude if chained indexes are used in place of their fixed base counterparts; the opposite will be true if there is *price bouncing behavior*;[16] i.e., if prices and quantities fluctuate erratically over time. Harvests of fresh fruits vary considerably for the same month of the year on a year over year basis, which leads to considerable fluctuations in prices and hence to price bouncing behavior. Thus for our empirical example, we found that the year over year monthly Laspeyres fixed base indexes exhibited a considerable amount of upward substitution bias and the chained Laspeyres indexes exhibited even more upward bias. On the other hand, the year over year monthly Paasche fixed base indexes exhibited a considerable amount of downward substitution bias and the chained Paasche indexes exhibited even more downward bias. Thus from the viewpoint of the economic approach to index number theory, the use of the Laspeyres and Paasche formulae is not recommended in the context of forming year over year monthly indexes.

From the viewpoint of the economic approach to index number theory, the bilateral Fisher and Törnqvist-Theil indexes have equally good properties; they are examples of *superlative index number formulae* and can deal adequately with substitution bias.[17] Moreover, they approximate each other numerically to the second order around any point that has equal price and quantity vectors in the two periods being compared.[18] Finally, the Fisher index has excellent properties from the viewpoint of the test approach to index number theory[19] and the Törnqvist-Theil index has excellent properties from the viewpoint of the stochastic approach to index number theory.[20]

[16] This term is due to Szulc (1983) (1987) who also demonstrated empirically the chain drift problem for the Laspeyres index when prices bounce.
[17] See Diewert (1976) who defined a superlative index number formula as one which was consistent with a wide variety of consumer substitution responses to changes in relative prices.
[18] See Diewert (1978).
[19] See Diewert (1992).
[20] See Theil (1967).

Thus these two indexes have very desirable properties from the perspective of a variety of approaches to index number theory. For the year over year monthly indexes listed in the Appendix, the fixed base Fisher and Törnqvist-Theil indexes approximated each other quite well for our empirical example.

From the viewpoint of the test approach to index number theory, the two fixed base superlative indexes have an advantage over their chained counterparts: they satisfy the following *multiperiod identity test*: if prices *and* quantities are the same in any two periods, the two fixed base indexes will register the same price level for these two periods. The two chained superlative indexes do not satisfy this identity test if there are four or more periods in the set of comparisons.[21]

The above considerations suggest that the two fixed base superlative indexes are preferred indexes in the above menu of eight possible indexes. However, there are two problems with the use of a fixed base index:

- The prices and quantities of the base period may not be representative of prices and quantities in subsequent periods.
- New products may appear and products present in the base period may disappear in subsequent periods making comparisons between distant periods difficult.

The second set of difficulties could be regarded as a special case of the first set of difficulties. In the context of our fresh fruit empirical example, the problem of new and disappearing products is not present. However, fluctuations in harvests certainly occurred and so there is the danger that the base period may not represent "typical" conditions and thus the choice of a different base period could lead to very different indexes.

In order to deal with the first difficulty above, we will turn to the use of *multilateral indexes*. Fisher was the first index number theorist to suggest a solution to the problem of fixed base price indexes defined over three or more periods being dependent on the choice of the base period. He suggested taking the *arithmetic average* of the T fixed base Fisher indexes that used each observation as the base period, if there are T periods in the comparison.[22] The resulting index is independent of the choice of a base period, or put differently, it treats all possible choices of a base period in a symmetric manner.[23]

Gini (1924) (1931) soon picked up on Fisher's idea and applied it to calculating relative price levels for several Italian cities but instead of taking an arithmetic average of the city specific fixed

[21] The corresponding strong identity test is: if *prices* are the same in any two periods, the multilateral index will register the same price level for these two periods. For materials on the test approach to multilateral index number theory, see Diewert (1988) (1999b) (2021b), Balk (1996) (2008), Zhang, Johansen and Nygaard (2019) and Diewert and Fox (2020).

[22] "There remains the practical question: if we are not going to use all six, what single curve is the best one to use in their place, for the general purpose of all comparisons over a series of years? Doubtless the very best as to accuracy, were it practicable, is the blend or average of all six. ... This is a compromise single series of six figures that can be substituted for the whole table of figures, for the purpose of blending all separate exact comparisons into one general *nearly* exact comparison." Irving Fisher (1922; 304-305). Fisher's T was equal to six.

[23] However, there are two disadvantages to the above multilateral approach to index number theory: (i) As new data become available, the multilateral indexes have to be recomputed and the prior indexes that applied to periods 1 to T are in general changed and (ii) not all bilateral comparisons between any two periods in the window of T observations are equally "good". These difficulties with the above multilateral methods can be overcome by using similarity linking which will be described below.

base Fisher indexes, he suggested taking the *geometric average* of the individual fixed base Fisher indexes. Eltetö and Köves (1964) and Szulc (1964) showed how Gini's multilateral index could derived as a solution to a least squares minimization problem and so the index is now referred to as the GEKS index. It should be noted that Balk (1980a) (1980b) (1980c) (1981) was a pioneer in applying multilateral indexes to seasonal data. However, he did not use the GEKS index in his empirical examples. Ivancic, Diewert and Fox (2011) suggested the use of the GEKS index in the time series context.

We now set up the notation that is required to describe how to calculate the year over year monthly GEKS indexes. Recall that the Laspeyres and Paasche indexes for month m in year y relative to year 1 were defined by definitions (2) and (3) above. In order to formally define the sequence of GEKS indexes for each month, we need to define the *Laspeyres* and *Paasche indexes* for month m in year y using month m in *year z* (instead of month m in *year 1*) as the base. These more general indexes, $P_L^m(y/z)$ and $P_P^m(y/z)$, are defined as follows:

(15) $P_L^m(y/z) \equiv \sum_{n\in S(m)} p_{y,m,n}q_{z,m,n}/\sum_{n\in S(m)} p_{z,m,n}q_{z,m,n}$; $\quad m = 1,...,M;\ y = 1,...,Y;\ z = 1,...,Y;$
(16) $P_P^m(y/z) \equiv \sum_{n\in S(m)} p_{y,m,n}q_{y,m,n}/\sum_{n\in S(m)} p_{z,m,n}q_{y,m,n}$; $\quad m = 1,...,1M;\ y = 1,...,Y;\ z = 1,...,Y.$

Thus for each month m, $P_L^m(y/z)$ compares the prices of available products in month m of year y in the numerator using the corresponding available products in month m of year z as weights to the prices of available products in month m of year z in the denominator using the corresponding available products in month m of year z as weights. For each month m, $P_P^m(y/z)$ compares the prices of available products in month m of year y in the numerator using the corresponding available products in month m of year y as weights to the prices of available products in month m of year z in the denominator again using the corresponding available products in month m of year y as weights. The corresponding *Fisher index* for month m in year y using month m in year z as the base, $P_F^m(y/z)$, is defined as the geometric mean of Laspeyres and Paasche indexes for month m in year y using month m in year z as the base period:

(17) $P_F^m(y/z) \equiv [P_L^m(y/z)P_P^m(y/z)]^{1/2}$; $\quad m = 1,...,M;\ y = 1,...,Y;\ z = 1,...,Y.$

The Fisher fixed base index for month m defined above by (4) chose month m in year 1 as the base period and formed the following sequence of year over year price levels relative to year 1: $P_F^m(1/1) = 1$, $P_F^m(2/1)$, $P_F^m(3/1)$, ..., $P_F^m(Y/1)$. But one could also use month m in year 2 as the base period and use the following sequence of price levels to measure year over year inflation for each month m: $P_F^m(1/2)$, $P_F^m(2/2) = 1$, $P_F^m(3/2)$, ..., $P_F^m(Y/2)$. Month m in each of Y years could be chosen as the base period and thus we end up with Y alternative series of Fisher price levels for each month. Since each of these sequences of price levels is equally plausible, the *GEKS price levels*, $p_{GEKS}^{y,m}$, for each month m for years y = 1,2,...,Y are defined as the geometric mean of the separate indexes we obtain by using each year as the base year:

(18) $p_{GEKS}^{y,m} \equiv [\prod_{z=1}^{Y} P_F^m(y/z)]^{1/Y}$; $\quad m = 1,...,M;\ y = 1,...,Y.$

Note that all time periods are treated in a symmetric manner in the above definitions. The *GEKS price indexes* $P_{GEKS}^{y,m}$ are obtained by normalizing the above price levels so that the period 1 index is equal to 1 for each month. Thus we have the following definitions for the month m year over year GEKS index for year y, $P_{GEKS}^{y,m}$:

(19) $P_{GEKS}^{y,m} \equiv p_{GEKS}^{y,m}/p_{GEKS}^{1,m}$; $\quad m = 1,...,M;\ y = 1,...,Y.$

If prices and quantities are the same in any two periods, then the resulting GEKS indexes will be identical for those two periods, which is a desirable property.

There is a problem associated with the use of the GEKS index in a time series context: when an additional month of data becomes available, the GEKS indexes need to be recomputed and the existing historical pattern of price levels will change in general. This poses problems for nonrevisable indexes like a Consumer Price Index. A solution to this problem was proposed by Ivancic, Diewert and Fox (2009) (2011). Their method added the price and quantity data for the most recent time period to a *window* of consecutive time periods and they also dropped the price and quantity data for the oldest period from the previous window of observations in order to obtain a new window. The GEKS indexes for the new window of observations were calculated in the usual way and the ratio of the index value for the last month in the new window to the index value for the previous month in the new window was used as an *update factor* for the value of the index for the last period in the existing index. The resulting indexes are called *Rolling Window GEKS indexes*. Unfortunately, the resulting indexes no longer satisfy the multiperiod identity test and so they are not entirely free of chain drift. However, empirical studies have shown that the method does not generate a substantial amount of chain drift. There is also a problem associated with exactly how should we link the latest data in the rolling window to the previously calculated indexes. Krsinich (2016; 383) called the above method for linking the new window to the previous window the *movement splice* method for linking the two windows. Krsinich (2016; 383) also suggested that a better choice to link the results of the new window to the previous window is to link the new observation to the index value in the second time period in the previous window of observations. She called this the *window splice* method. Let T be the length of the window. De Haan (2015; 26) suggested that the link period t should be chosen to be in the middle of the first window time span; i.e., choose t = T/2 if T is an even integer or t = (T+1)/2 if T is an odd integer. The Australian Bureau of Statistics (2016; 12) called this the *half splice* method for linking the results of the two windows. Diewert and Fox (2020) suggested linking the last observation in the current window to all possible choices of periods that overlap in the two windows and taking the geometric mean of the resulting estimates for the price level in the final period of the current window. They termed this the *mean splice* and they recommended it as perhaps being best since the result of choosing each of the possible linking periods is equally valid.[24]

For our empirical example, we simply implemented the GEKS method using the entire 6 years of data for each month; i.e., we did not calculate rolling window GEKS indexes. Thus these estimated year over year GEKS indexes listed in Table A.21 for each month are not practical real time indexes but they are of interest so that the effects of changing the base year can be studied. We will discuss how $P_{GEKS}{}^{y,m}$ defined by (19) performed using our Israeli data set on strongly seasonal fresh fruits after we have defined some alternative multilateral indexes.

The GEKS multilateral method treats each set of price indexes using the prices of one period as the base period as being equally valid and hence an averaging of the resulting parities seems to be appropriate under this hypothesis. Thus the method is "democratic" in that each bilateral index number comparison between any two periods gets the same weight in the overall method. However, *it is not the case that all bilateral comparisons of price between two periods are equally accurate*: if the relative prices in periods r and t are very similar, then the Laspeyres and Paasche price indexes will be very close to each other and hence it is likely that the "true" price comparison between these two periods will be very close to the bilateral Fisher index between

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[24] This method for linking the two windows was also suggested by Ivancic, Diewert and Fox (2011; 33) in a footnote. Later in this chapter when we study similarity linking, we will see that all links are not necessarily equally good.

these two periods. In particular, if the two price vectors are exactly proportional, then we want the price index between these two periods to be equal to the factor of proportionality and the direct Fisher index between these two periods satisfies this proportionality test. On the other hand, the GEKS index comparison between the two periods would not in general satisfy this proportionality test.[25] Furthermore if prices are identical between two periods but the quantity vectors are different, then the GEKS price index between the two periods would not equal unity in general.[26]

Linking observations that have the most *similar structure of relative prices* addresses these difficulties with the GEKS method. Hill (1997) (1999a) (1999b) (2009) and Diewert (2009) developed this multilateral similarity linking method in the context of making cross country comparisons. In the time series context, this linking of observations with the most similar price structures was pioneered by Hill (2001) (2004).

A key aspect of this methodology is the choice of the measure of similarity (or dissimilarity) of the relative price structures of two observations. Various measures of the similarity or dissimilarity of relative price structures have been proposed by Allen and Diewert (1981), Kravis, Heston and Summers (1982; 104-106), Hill (1997) (2009), Aten and Heston (2009) and Diewert (2009). However, Hill and Timmer (2006) pointed out a problem with these measures of relative price dissimilarity: they do not take into account the *lack of matching problem*; i.e., these measures fail to recognize that bilateral comparisons of prices made over a smaller number of products are not as reliable as comparisons made over a larger number of matched products.[27] This lack of matching problem is a big one in the context of constructing index numbers for a product category where many or most products are only available in some months of the year. In our empirical example, only about 60% of the seasonal products are available in a typical month.

For our empirical example, we will use the *predicted share measure of relative price dissimilarity*. In situations where carry forward prices are not used, this method penalizes a lack of price matching between two observations.[28] In order to define this measure, it is useful to introduce some notation for the *vectors* of prices and quantities for month m in year y, $p^{y,m}$ and $q^{y,m}$. If product n in month m of year y is present, then define the price and quantity of that product to be $p_{y,m,n}$ and $q_{y,m,n}$, as usual. If product n in month m of year y is not present, then define the quantity of that product to be 0 so that $q_{y,m,n} \equiv 0$ and define $p_{y,m,n}$ to be the year over year carry forward (or backward) price. With these additional variables defined, the N dimensional price and quantity vectors for month m in year y are well defined as $p^{y,m} \equiv [p_{y,m,1},p_{y,m,2},...,p_{y,m,N}]$ and $q^{y,m} \equiv [q_{y,m,1},q_{y,m,2},...,q_{y,m,N}]$ for y = 1,...,Y and m = 1,...,M. With this new notation, prices and quantities

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[25] If both prices and quantities are proportional to each other for the two periods being compared, then the GEKS price index between the two periods will satisfy this (weak) proportionality test. However, we would like the GEKS price index between the two periods to satisfy the strong proportionality test; i.e., if the two price vectors are proportional (and the two quantity vectors are not necessarily proportional to each other), then we would like the GEKS price index between the two periods to equal the factor of proportionality.
[26] See Zhang, Johansen and Nygaard (2019; 689) on this point.
[27] "Although these measures perform well when there are few gaps in the data, they can generate highly misleading results when there are many gaps. This is because they fail to penalize bilateral comparisons made over a small number of matched headings." Robert Hill and Marcel Timmer (2006; 366). Hill and Timmer go on and propose a measure of relative price dissimilarity that penalizes a lack of price matching. Their measure is based on econometric considerations. The measure that we use also penalizes a lack of price matching but it has a different motivation.
[28] In the present section where year over year carry forward prices are used, all prices are matched so there is no penalty for a lack of matching. However, in the next section, we will not use any form of imputed price so the predicted share measure of price dissimilarity will penalize a lack of matching.

are well defined for all N products for each year and month. Thus *the expenditure share for product n in month m and year y*, $s_{y,m,n}$, can now be defined for all N products as:

(20) $s_{y,m,n} \equiv p_{y,m,n}q_{y,m,n}/p^{y,m}\cdot q^{y,m}$; $\quad$ y = 1,...,Y; m = 1,2,...,M ; n = 1,2,...,N

where $p^{y,m}\cdot q^{y,m} \equiv \Sigma_{n=1}^{N} p_{y,m,n}q_{y,m,n}$ is the inner product of the vectors $p^{y,m}$ and $q^{y,m}$. Note that even though these expenditure shares use imputed prices for missing products, they are equal to the actual expenditure shares for all products.

Now think of using the *prices* of month m in year z and the *quantities* of month m in year y to *predict* the actual month m, year y, product n expenditure share $s_{y,m,n}$ defined by (20) for n = 1,...,N. Denote this *predicted share* by $s_{z,y,m,n}$, which is defined as follows:

(21) $s_{z,y,m,n} \equiv p_{z,m,n}q_{y,m,n}/p^{z,m}\cdot q^{y,m}$; $\quad$ y = 1,...,Y; z = 1,...,Y; m = 1,2,...,M ; n = 1,2,...,N.

If the prices in month m of year y are proportional to the prices of month m in year z so that $p^{z,m} = \lambda p^{y,m}$ where λ is a positive number, then it can verified that the predicted shares defined by (21) will be equal to the actual expenditure shares defined by (20) for month m in year y; i.e., for the two months defined by y,m and z,m, we will have $s_{y,m,n} = s_{z,y,m,n}$ for n = 1,...,N. The following *Predicted Share measure of relative price dissimilarity* between the prices of month m in year y and the prices of month m in year z, $\Delta_{PS}(p^{z,m},p^{y,m},q^{z,m},q^{y,m})$, is well defined even if some product prices and shares in the two months being compared are equal to zero:

$$(22)\ \Delta_{PS}(p^{z,m},p^{y,m},q^{z,m},q^{y,m}) \equiv \Sigma_{n=1}^{N} [s_{y.m,n} - s_{z,y,m,n}]^2 + \Sigma_{n=1}^{N} [s_{z.m,n} - s_{y,z,m,n}]^2$$
$$= \Sigma_{n=1}^{N} [(p_{y,m,n}q_{y,m,n}/p^{y,m}\cdot q^{y,m}) - (p_{z,m,n}q_{y,m,n}/p^{z,m}\cdot q^{y,m})]^2$$
$$+ \Sigma_{n=1}^{N} [(p_{z,m,n}q_{z,m,n}/p^{z,m}\cdot q^{z,m}) - (p_{y,m,n}q_{z,m,n}/p^{y,m}\cdot q^{z,m})]^2.$$

In general, $\Delta_{PS}(p^r,p^t,q^r,q^t)$ takes on values between 0 and 2. If $\Delta_{PS}(p^r,p^t,q^r,q^t) = 0$, then it must be the case that relative prices are the same in month m of years z and y; i.e., we have $p^{z,m} = \lambda p^{y,m}$ for some $\lambda > 0$. A bigger value of $\Delta_{PS}(p^r,p^t,q^r,q^t)$ generally indicates bigger deviations from price proportionality.

To see how this predicted share measure of relative price dissimilarity turned out for our Israeli data on 14 classes of fresh fruits for the month of January, see Table 1 below. The month m is equal to 1 (January). The years y and z range from 1 to 6. Fruits 1, 2, 4, 5, 6, 12 and 13 were always available in January for each of the six years in our sample; the other 7 fruits were always missing in January. Thus there are no carry forward imputed prices that are used for the January data. For a listing of the nonzero price $p_{y,1,n}$ and quantity $q_{y,1,n}$ data for January 2012-2017 (years 1-6), see Table A.1 in the Appendix.

Table 1: Predicted Share Measures of Price Dissimilarity for January for Years 1-6

m = 1	**y = 1**	**y = 2**	**y = 3**	**y = 4**	**y = 5**	**y = 6**
z = 1	**0.00000**	**0.00306**	**0.00632**	**0.00062**	**0.00810**	**0.00363**
z = 2	**0.00306**	**0.00000**	**0.00082**	**0.00429**	**0.00325**	**0.00119**
z = 3	**0.00632**	**0.00082**	**0.00000**	**0.00696**	**0.00375**	**0.00233**
z = 4	**0.00062**	**0.00429**	**0.00696**	**0.00000**	**0.01019**	**0.00421**
z = 5	**0.00810**	**0.00325**	**0.00375**	**0.01019**	**0.00000**	**0.00171**
z = 6	**0.00363**	**0.00119**	**0.00233**	**0.00421**	**0.00171**	**0.00000**

The matrix of predicted share measures of relative price dissimilarity for the month of January for all pairs of years in our sample is nonnegative, symmetric and has zeros down its main diagonal. The measure of relative price dissimilarity between years 1 and 2 is 0.00306, between years 1 and 3 is .00632 and so on.

The above matrix is used to construct $P_S^{y,1}$, the *similarity linked price index for January*. The *real time version* of this index is constructed as follows. Set $P_S^{1,1} \equiv 1$. The year over year index for January in year 2 is set equal to the bilateral Fisher index $P_F^m(y/z)$ where m = 1, y = 2 and z = 1 (see definition (17) above). Using our new vector notation, this Fisher index is equal to $[p^{2,1}\cdot q^{1,1}\ p^{2,1}\cdot q^{2,1}/\ p^{1,1}\cdot q^{1,1}\ p^{1,1}\cdot q^{2,1}]^{1/2}$. Thus the year 2 similarity linked index for January is $P_S^{2,1} \equiv P_F^1(2/1)$. Now look down the y = 3 column in Table 1. We need to link year 3 to either year 1 or year 2. The dissimilarity measures for these two years are 0.00632 and 0.00082 respectively. The degree of relative price dissimilarity is far smaller for the link to year 2 than it is to year 1 (year 3 January prices are much closer to being proportional to year 2 prices than to year 1 prices) so we use the Fisher link from period 2 to period 3, $P_F^1(3/2)$. Thus the final year 3 similarity linked index for January is $P_S^{3,1} \equiv P_S^{2,1}\times P_F^1(3/2)$. Now we need to link year 4 to either year 1, 2 or 3. Look down the y = 4 column in Table 1 to find the lowest dissimilarity measure above the main diagonal of the matrix. The smallest of the 3 numbers 0.00062, 0.00429 and 0.00696 is 0.00062. Thus we link the year 4 January data to the year 1 January data using the Fisher January link from year 1 to year 4, $P_F^1(4/1)$, and the year 4 similarity linked final index value is $P_S^{4,1} \equiv P_S^{1,1}\times P_F^1(4/1) = P_F^1(4/1)$. *Thus for each year, as the new January data become available, we use the Fisher bilateral index that links the new period to the previous period that has the lowest measure of relative price dissimilarity*. The final two bilateral links are year 5 to year 2 and year 6 to year 2. The resulting year 5 and 6 similarity linked index values are $P_S^{5,1} \equiv P_S^{2,1}\times P_F^1(5/2)$ and $P_S^{6,1} \equiv P_S^{2,1}\times P_F^1(6/2)$. The optimal set of *bilateral links* for the January year over year real time similarity linked indexes can be summarized as follows:

$$\begin{tikzcd} & 5 \arrow[d, dash] & \\ 1 \arrow[r, dash] \arrow[d, dash] & 2 \arrow[r, dash] \arrow[d, dash] & 3 \\ 4 & 6 & \end{tikzcd}$$

Using our empirical data set, we calculated the 10 year over year alternative indexes for January that are defined above. These indexes are the fixed base Laspeyres, Paasche, Fisher and Törnqvist Theil indexes, $P_{LFB}^{y,1}$, $P_{PFB}^{y,1}$, $P_{FFB}^{y,1}$ and $P_{TFB}^{y,1}$, the corresponding chained indexes, $P_{LCH}^{y,1}$, $P_{PCH}^{y,1}$, $P_{FCH}^{y,1}$ and $P_{TCH}^{y,1}$, the GEKS index, $P_{GEKS}^{y,1}$, and the predicted share similarity linked index, $P_S^{y,1}$. The year superscript y takes on the values 1-6. These indexes are listed in Table 2 below.

Table 2: Year over Year Alternative Indexes for January

Year y	$P_{LFB}^{y,1}$	$P_{PFB}^{y,1}$	$P_{FFB}^{y,1}$	$P_{TFB}^{y,1}$	$P_{LCH}^{y,1}$	$P_{PCH}^{y,1}$	$P_{FCH}^{y,1}$	$P_{TCH}^{y,1}$	$P_{GEKS}^{y,1}$	$P_S^{y,1}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.99746	0.99881	0.99813	0.99817	0.99746	0.99881	0.99813	0.99817	0.99814	0.99813
3	1.03276	1.01894	1.02583	1.02591	1.02762	1.01799	1.02280	1.02261	1.02295	1.02280
4	1.01159	1.00992	1.01076	1.01072	1.01586	0.99872	1.00725	1.00700	1.00816	1.01076
5	1.12212	1.12896	1.12554	1.12582	1.14808	1.10989	1.12883	1.12854	1.12973	1.13415
6	1.07410	1.06543	1.06976	1.06889	1.09958	1.04827	1.07362	1.07252	1.07153	1.06944
Mean	1.03970	1.03700	1.03830	1.03830	1.04810	1.02890	1.03840	1.03810	1.03840	1.03920

Looking at Table 2, it can be seen that the fixed base Laspeyres indexes exceed the fixed base Paasche indexes by about 0.27 percentage points on average. The gap between the chained Laspeyres indexes and the chained Paasche indexes is much larger at 1.92 percentage points. These gaps indicate that the Laspeyres and Paasche indexes suffer from some upward or downward substitution bias. The larger gap for the chained indexes also indicates that the chained Laspeyres and Paasche indexes are subject to a considerable amount of chain drift. The remaining 6 indexes are all close to each other on average.

Our year over year data on January fresh fruit consumption for Israel for the 6 years in our sample did not have any missing products that changed from year to year; fruits 1, 2, 4, 5, 6, 12 and 13 were always available in January for each of the six years in our sample; the other 7 fruits were always missing in January. Thus no imputed prices were used for the January data. However, imputed carry forward (or backward) prices were used for other months.

For example, for our particular data set, the month of May has 8 missing prices which were imputed by 6 carry forward prices and 2 carry backward prices. Products 1, 2, 3, 5, 6, 7 and 10 were always present in May. Products 4, 11, 12 and 14 were always missing in May. The remaining products 8, 9 and 13 were sometimes present in May and were sometimes absent. Thus carry forward or carry backward prices were used to impute the missing prices for products 8, 9 and 13. The data for May are listed in Tables A.7 and A.8 in the Appendix. The 8 imputed prices are listed in these tables using italics. To see how the predicted share measure of relative price dissimilarity defined by (22) turned out for our Israeli data for the month of May, see Table 3 below. The month m is equal to 5 (May). As usual, the years y and z range from 1 to 6.

Table 3: Predicted Share Measures of Price Dissimilarity for May for Years 1-6

m = 5	**y = 1**	**y = 2**	**y = 3**	**y = 4**	**y = 5**	**y = 6**
z = 1	**0.00000**	**0.00617**	**0.00250**	**0.02222**	**0.01103**	**0.01324**
z = 2	**0.00617**	**0.00000**	**0.00578**	**0.02768**	**0.00883**	**0.01908**
z = 3	**0.00250**	**0.00578**	**0.00000**	**0.01226**	**0.00409**	**0.00690**
z = 4	**0.02222**	**0.02768**	**0.01226**	**0.00000**	**0.01060**	**0.00175**
z = 5	**0.01103**	**0.00883**	**0.00409**	**0.01060**	**0.00000**	**0.00810**
z = 6	**0.01324**	**0.01908**	**0.00690**	**0.00175**	**0.00810**	**0.00000**

The *real time* set of bilateral links which minimize the predicted share measure of relative price dissimilarity for the May data for the current year with the May data for a prior year are as follows: link 2 to 1; 3 to 1; 4 to 3; 5 to 3 and 6 to 4. The optimal set of links can be summarized as follows:

```
1 – 2
|
3 – 4 – 6
    |
    5.
```

Using the price and quantity data for May that is listed in Tables A.7 and A.8 in the Appendix, we calculated the May year over year indexes using the fixed base Laspeyres, Paasche, Fisher and Törnqvist Theil indexes, $P_{LFB}^{y,5}$, $P_{PFB}^{y,5}$, $P_{FFB}^{y,5}$ and $P_{TFB}^{y,5}$, the corresponding chained indexes, $P_{LCH}^{y,5}$, $P_{PCH}^{y,5}$, $P_{FCH}^{y,5}$ and $P_{TCH}^{y,5}$, the GEKS index, $P_{GEKS}^{y,5}$, and the predicted share similarity linked index, $P_S^{y,5}$ for the years 1-6. These indexes are listed in Table 4 below.

Table 4: Year over Year Alternative Indexes for May

Year y	$P_{LFB}^{y,5}$	$P_{PFB}^{y,5}$	$P_{FFB}^{y,5}$	$P_{TFB}^{y,5}$	$P_{LCH}^{y,5}$	$P_{PCH}^{y,5}$	$P_{FCH}^{y,5}$	$P_{TCH}^{y,5}$	$P_{GEKS}^{y,5}$	$P_{S}^{y,5}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.95731	0.91814	0.93752	0.93708	0.95731	0.91814	0.93752	0.93708	0.93879	0.93752
3	1.04955	1.02931	1.03938	1.03929	1.07750	0.99674	1.03634	1.03544	1.04223	1.03938
4	1.29576	1.26861	1.28211	1.27958	1.34446	1.21671	1.27899	1.27733	1.28376	1.28275
5	1.15686	1.15394	1.15540	1.15718	1.22628	1.06571	1.14318	1.14348	1.15227	1.14281
6	1.29885	1.29900	1.29893	1.29611	1.36519	1.18589	1.27239	1.27244	1.29548	1.29399
Mean	1.12640	1.11150	1.11890	1.11820	1.16180	1.06390	1.11140	1.11100	1.11880	1.11610

The results for the year over year May indexes are similar to the results for the year over year January indexes in some respects:

- The chained Laspeyres indexes $P_{LCH}^{y,5}$ ended up at 1.36519 which is well above the final value for the chained Paasche indexes $P_{PCH}^{y,5}$ which was 1.18589;
- The fixed base Fisher and Törnqvist-Theil indexes, the GEKS indexes and the similarity linked indexes, $P_{FFB}^{y,5}$, $P_{TFB}^{y,5}$, $P_{GEKS}^{y,5}$ and $P_{S}^{y,5}$, all ended up at much the same levels and in general, were quite close to each other.

The big difference between the May results and the January results is that the chained Fisher and Törnqvist-Theil indexes for May, $P_{FCH}^{y,5}$, $P_{TCH}^{y,5}$, ended up well below the other May superlative indexes, $P_{FFB}^{y,5}$, $P_{TFB}^{y,5}$, $P_{GEKS}^{y,5}$ and $P_{S}^{y,5}$. This is due to the influence of the 6 carry forward prices that are used in the May year over year data. There were no imputed prices for the January data and hence there was no carry forward bias for this month. Thus if there is general inflation in the segment of the economy under consideration and carry forward prices are used to replace missing prices, then the use of chained superlative indexes will tend to lead to indexes that are biased downwards relative to their fixed base counterparts.

The year over year indexes for all 12 months are reported in Table A.21 in the Appendix. The following table reports the overall mean and variance of all 10 indexes, where the index values are stacked into a single column with 72 rows for each of the 10 indexes.

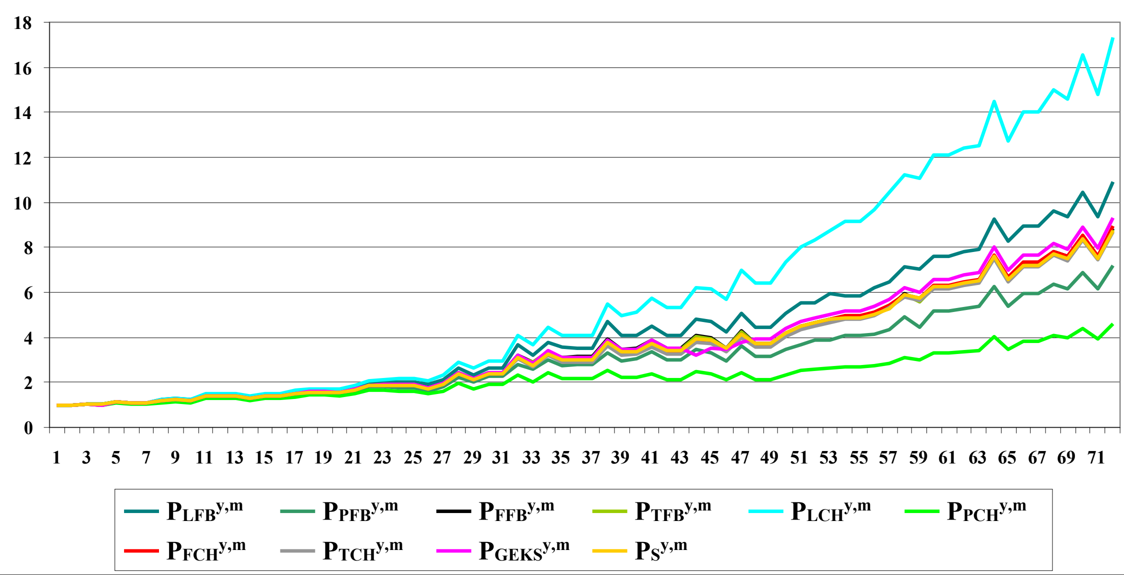
Table 5: Year over Year Index Means and Variances Over All Months and Years for Ten Indexes Using Carry Forward Prices

	$P_{LFB}^{y,m}$	$P_{PFB}^{y,m}$	$P_{FFB}^{y,m}$	$P_{TFB}^{y,m}$	$P_{LCH}^{y,m}$	$P_{PCH}^{y,m}$	$P_{FCH}^{y,m}$	$P_{TCH}^{y,m}$	$P_{GEKS}^{y,m}$	$P_{S}^{y,m}$
Mean	1.1365	1.1001	1.1180	1.1170	1.1560	1.0817	1.1176	1.1154	1.1111	1.1178
Variance	0.0161	0.0101	0.0125	0.0123	0.0203	0.0079	0.0121	0.0117	0.0130	0.0122

On average, the cumulated year over year fixed base Laspeyres indexes $P_{LFB}^{y,m}$ exceeded their cumulated fixed base Paasche counterparts by $1.1365 - 1.1001 = 0.0364$ or 3.64 percentage points. The average gap between the chained Laspeyres and Paasche indexes was $1.1560 - 1.0817 = 0.0743$ or 7.43 percentage points. These are substantial differences and indicate that the use of these indexes should be avoided. The fixed base Fisher, fixed base Törnqvist Theil, chained Fisher and Predicted Share similarity linked indexes, $P_{FFB}^{y,m}$, $P_{TFB}^{y,m}$, $P_{FCH}^{y,m}$ and $P_{S}^{y,m}$ all had similar means and variances and performed equally well on our particular data set with means between 1.117 and 1.118. The mean of the chained Törnqvist Theil indexes was a bit lower at 1.1154 and their variance was also lower. This may reflect the fact that chaining indexes that use carry forward prices in a period of high general inflation will tend to lower the average inflation rate and may also lower the variance. The mean of the GEKS indexes was 1.111 which is below 1.117. On the other hand, the variance of the GEKS indexes was 0.0130 which is above the range of the variances for the "best" indexes, $P_{FFB}^{y,m}$, $P_{TFB}^{y,m}$, $P_{FCH}^{y,m}$ and $P_S^{y,m}$, which was between 0.0121 and 0.0125.

In order to illustrate the differences between the ten different index number formulae, we cumulated the year over year indexes listed in Table A.21 in the Appendix and plotted the resulting cumulated indexes on Chart 1 below. Thus the first 6 points for the series P_{LFB} are the January year over year fixed base Laspeyres indexes for years 1- 6: $P_{LFB}^{1,1}$, $P_{LFB}^{2,1}$, ..., $P_{LFB}^{6,1}$. The next 6 points for the P_{LFB} series are the February year over year fixed base Laspeyres indexes for years 1-6 times the final value for the January fixed base Laspeyres series, $P_{LFB}^{6,1}$. Thus the values for the listed P_{LFB} series on Chart 1 for observations 7-12 are the cumulated indexes $P_{LFB}^{6,1}\times P_{LFB}^{1,2}$, $P_{LFB}^{6,1}\times P_{LFB}^{2,2}$, ..., $P_{LFB}^{6,1}\times P_{LFB}^{6,2}$. The next 6 points for the P_{LFB} series are the March year over year fixed base Laspeyres indexes for years 1-6 times the cumulated value for observation 12 of the cumulated fixed base Laspeyres series, which is $P_{LFB}^{6,1}\times P_{LFB}^{6,2}$. Thus the values for the listed P_{LFB} series on Chart 1 for observations 13-18 are the cumulated indexes $P_{LFB}^{6,1}\times P_{LFB}^{6,2}\times P_{LFB}^{1,3}$, $P_{LFB}^{6,1}\times P_{LFB}^{6,2}\times P_{LFB}^{2,3}$, ..., $P_{LFB}^{6,1}\times P_{LFB}^{6,2}\times P_{LFB}^{6,3}$. And so on. The final 6 observations for the P_{LFB} series are defined as $P_{LFB}^{6,1}\times P_{LFB}^{6,2}\times P_{LFB}^{6,3}\times \ldots\times P_{LFB}^{6,11}$ times the December year over year fixed base Laspeyres indexes for years 1-6, $P_{LFB}^{1,12}$, $P_{LFB}^{2,12}$, ..., $P_{LFB}^{6,12}$. The remaining nine cumulated series were constructed in a similar manner.

Chart 1: Cumulated Year over Year Indexes using Carry Forward Prices



The highest series is the cumulated chained Laspeyres index P_{LCH} followed by the cumulated fixed base Laspeyres index, P_{LFB}. The lowest series is the cumulated chained Paasche index P_{PCH} followed by the cumulated fixed base Paasche index, P_{PFB}. The remaining 6 indexes are all clustered together in the middle of these outlier series, with the cumulated GEKS indexes P_{GEKS} lying slightly above the remaining 5 clustered indexes. The cumulated chained Törnqvist Theil indexes P_{TCH} are just a bit below the other 4 clustered indexes.

The above series used carry forward or carry backward prices for seasonal products which were at times not available in their "regular" seasonally available months. However, when there is general inflation (or deflation) in an economy, there is a risk of introducing a significant amount of bias when carry forward prices are used to fill in for the missing prices. Hence in the following section, we will calculate year over year indexes without using carry forward prices.

Once the Laspeyres and Paasche indexes are eliminated from consideration, it can be seen that the remaining 6 year over year monthly indexes are all fairly close to each other.

In the following section, we will construct the same 10 indexes but we will not use any imputed prices. Instead, we will use bilateral indexes that are based on the common set of products that are actually present in both periods for each bilateral comparison. The resulting indexes can then be compared with the indexes that are plotted in Chart 1 above. The new indexes which do not use carry forward prices are listed in Table A.22 in the Appendix.

We conclude this section with a brief discussion on the use of carry forward prices by statistical agencies. In many cases, a simple carry forward price for a missing price is not used; instead the price of a close substitute is used or an *inflation adjusted carry forward price* is used. In the latter case, the last available price is multiplied by an index of prices for related products that are available in the two periods that are being compared.[29] Depending on the price index concept that is being used, the use of inflation adjusted carry forward prices will be at least approximately equivalent to simply using the index that is restricted to the products that are available in the two periods under consideration. Thus in the following section, we will look at the use of *maximum overlap bilateral indexes*; i.e., products that are not present in both periods being compared are simply dropped. The problem with using the price of a close substitute to fill in a missing price is that the choice of the substitute product is necessarily somewhat arbitrary. To eliminate this arbitrariness, we will focus on the construction of maximum overlap indexes (or matched model indexes) in the following section.

3. Maximum Overlap Year over Year Monthly Indexes

Recall the notation that was introduced in section 1 where the set of commodities n which are present in the marketplace during month m of year y was denoted by S(y,m). Data on prices and quantities are available for Y years and say M = 12 months. Again the price of commodity n in month m of year y is denoted by $p_{y,m,n}$ and the corresponding quantity is denoted by $q_{y,m,n}$. In the present section, we do not use carry forward prices so if product n is missing in month m of year y, we set $p_{y,m,n} = 0$ and $q_{y,m,n} = 0$. Using these new prices and quantities, the *expenditure share for product n in month m and year y*, $s_{y,m,n}$, can now be defined for all N products as:[30]

$$(23)\quad s_{y,m,n} \equiv p_{y,m,n}q_{y,m,n}/p^{y,m}\cdot q^{y,m}\ ; \qquad y = 1,\ldots,Y;\ m = 1,2,\ldots,M\ ;\ n = 1,2,\ldots,N.$$

In the previous section, the Laspeyres, Paasche and Fisher indexes that compared the prices of month m in year y to the prices of month m in year z were defined by (15)-(17). These definitions used carry forward and carry backward prices for prices of seasonal commodities which happened to be absent in some years. In the present section, we want to avoid the use of any

[29] Armknecht and Maitland-Smith (1999) have a good discussion of the various methods used by statistical agencies to construct some sort of inflation adjusted carry forward price. This discussion is very relevant in recent times when Covid problems substantially increased the frequency of missing prices.

[30] These "new" expenditure shares turn out to be identical to the expenditure shares defined by (20) in the previous section.

imputed prices so these indexes are redefined by definitions (24)-(26) below for m = 1,...,M; y = 1,...,Y; z = 1,...,Y:

(24) $P_L^{m*}(y/z) \equiv \sum_{n\in S(y,m)\cap S(z,m)} p_{y,m,n}q_{z,m,n}/\sum_{n\in S(y,m)\cap S(z,m)} p_{z,m,n}q_{z,m,n}$;
(25) $P_P^{m*}(y/z) \equiv \sum_{n\in S(y,m)\cap S(z,m)} p_{y,m,n}q_{y,m,n}/\sum_{n\in S(y,m)\cap S(z,m)} p_{z,m,n}q_{y,m,n}$.;
(26) $P_F^{m*}(y/z) \equiv [P_L^{m*}(y/z)P_P^{m*}(y/z)]^{1/2}$.

The indexes defined by (24)-(26) are called *bilateral maximum overlap Laspeyres, Paasche and Fisher indexes* respectively.[31] The Laspeyres index that compares the prices of month m in year y to the prices of month m in year z, $P_L^{m*}(y/z)$, compares the prices of month m products that are available *in both year y and year z*. The jointly available product prices of year y appear in the numerator and are compared to the jointly available products of year z which appear in the denominator. The quantities of jointly available products for year z appear as weights in both numerator and denominator. Similarly, the Paasche index that compares the prices of month m in year y to the prices of month m in year z, $P_P^{m*}(y/z)$, compares the prices of month m products that are available *in both year y and year z*. The jointly available product prices of year y appear in the numerator and are compared to the jointly available products of year z which appear in the denominator. The quantities of jointly available products for year y appear as weights in both numerator and denominator. As usual, the corresponding Fisher index $P_F^{m*}(y/z)$ is the geometric mean of $P_L^{m*}(y/z)$ and $P_P^{m*}(y/z)$.

The sequence of maximum overlap year over year fixed base Laspeyres indexes for month m will be denoted by $P_{LFB}^{y,m*}$ for y = 1,2,...,Y. For our empirical example, Y = 6 and the *year over year maximum overlap fixed base Laspeyres indexes* $P_{LFB}^{y,m*}$ for months m = 1,...,12 are defined to be the indexes $P_L^{m*}(1/1)$, $P_L^{m*}(2/1)$, $P_L^{m*}(3/1)$, $P_L^{m*}(4/1)$, $P_L^{m*}(5/1)$, $P_L^{m*}(6/1)$ where the maximum overlap Laspeyres link indexes $P_L^{m*}(y/z)$ are defined by (24). Similarly, the *year over year maximum overlap fixed base Paasche indexes* $P_{PFB}^{y,m*}$ for months m = 1,...,12 are defined to be the indexes $P_P^{m*}(1/1)$, $P_P^{m*}(2/1)$, $P_P^{m*}(3/1)$, $P_P^{m*}(4/1)$, $P_P^{m*}(5/1)$, $P_P^{m*}(6/1)$ where the maximum overlap Paasche link indexes $P_P^{m*}(y/z)$ are defined by (25). Finally, the *year over year maximum overlap fixed base Fisher indexes* $P_{FFB}^{y,m*}$ for months m = 1,...,12 are defined to be the indexes $P_F^{m*}(1/1)$, $P_F^{m*}(2/1)$, $P_F^{m*}(3/1)$, $P_F^{m*}(4/1)$, $P_F^{m*}(5/1)$, $P_F^{m*}(6/1)$ where the maximum overlap Fisher bilateral link indexes $P_F^{m*}(y/z)$ are defined by (26). These fixed base maximum overlap Laspeyres, Paasche and Fisher indexes for our May data are listed below in Table 6.[32]

Define the *year over year maximum overlap chained Laspeyres, Paasche* and *Fisher indexes* for month m in year 1, $P_{LCH}^{1,m*}$, $P_{PCH}^{1,m*}$ and $P_{FCH}^{1,m*}$ as unity:

(27) $P_{LCH}^{1,m*} \equiv 1$; $P_{PCH}^{1,m*} \equiv 1$; $P_{FCH}^{1,m*} \equiv 1$; m = 1,....,M.

For years following year 1, the above maximum overlap indexes for the same month m are defined by cumulating the corresponding successive annual year over year links defined by (24)-(26); i.e., we have the following definitions:

(28) $P_{LCH}^{y,m*} \equiv P_{LCH}^{y-1,m*}\, P_L^{m*}(y/(y-1))$; m = 1,...,M; y = 2,...,Y;

[31] The idea of restricting bilateral index number comparisons of prices in two periods to the set of prices of products that are present in both periods can be traced back to Keynes (1909) (1930; 94).
[32] If the set of available seasonal products is the same every year for a particular month, then the maximum overlap indexes for that month will coincide with the corresponding indexes defined in the previous section, since there are no imputed prices for the year over year indexes when the available products are the same every year for the given month.

(29) $P_{PCH}{}^{y,m*} \equiv P_{PCH}{}^{y-1,m*} P_P{}^{m*}(y/(y-1))$; $m = 1,...,M; y = 2,...,Y;$
(30) $P_{FCH}{}^{y,m*} \equiv P_{FCH}{}^{y-1,m*} P_F{}^{m*}(y/(y-1))$; $m = 1,...,M; y = 2,...,Y.$

The *maximum overlap GEKS price levels*, $p_{GEKS}{}^{y,m*}$, for each month m for years y = 1,2,...,Y is defined as the geometric mean of the separate indexes we obtain by using each year as the base year:

(31) $p_{GEKS}{}^{y,m*} \equiv [\prod_{z=1}^{Y} P_F{}^{m*}(y/z)]^{1/Y}$; $m = 1,...,M; y = 1,...,Y$

where $P_F{}^{m*}(y/z)$ is defined by (26). The *maximum overlap GEKS price indexes* $P_{GEKS}{}^{y,m*}$ are obtained by *normalizing* the above price levels so that the period 1 index is equal to 1. Thus we have the following definitions for the month m *year over year maximum overlap GEKS index* for year y, $P_{GEKS}{}^{y,m*}$:

(32) $P_{GEKS}{}^{y,m*} \equiv p_{GEKS}{}^{y,m*}/p_{GEKS}{}^{1,m*}$; $m = 1,...,M; y = 1,...,Y.$

The maximum overlap GEKS indexes along with the chained maximum overlap Laspeyres, Paasche and Fisher indexes for our May data are listed below in Table 6.

Constructing the bilateral maximum overlap Törnqvist Theil index between every pair of years using the data for month m is more complicated. It is necessary to construct *conditional expenditure shares* which are expenditure shares for product n for month m in year y that are conditional on product n being purchased in both years y and z. First, we note that $q_{y,m,n}$ is well defined for all y,m and n as actual expenditures on product n for month m in year y. If there is no expenditure on product n for month m in year y, $q_{y,m,n}$ is defined to be equal to 0. In the 0 expenditure case, define the corresponding price, $p_{y,m,n}$, to be 0 as well. In the case where $q_{y,m,n} > 0$, then the corresponding price $p_{y,m,n}$ is defined to be the usual positive unit value price. With these conventions, $p_{y,m,n}$ and $q_{y,m,n}$ are defined for all y, m and n. Now define the *expenditure for product n in month m of year y, conditional on positive month m, year z quantities*, $e_{y,z,m,n}$, as follows:[33]

(33) $e_{y,z,m,n} \equiv p_{y,m,n}q_{y,m,n}$ if $q_{z,m,n} > 0$; $y = 1,...,Y; z = 1,...,Y; m = 1,...,M; n = 1,...,N;$
$\equiv 0$ if $q_{z,m,n} = 0$.

Thus $e_{y,z,m,n}$ will be positive if and only if there are sales of product n in month m for years y and z. Define the *total expenditure on products sold in month m of year y conditional on positive year z expenditure on products sold in month m of year z*, $e_{y,z,m}$, as the sum over n of the $e_{y,z,m,n}$ defined by (33):

(34) $e_{y,z,m} \equiv \Sigma_{n=1}^{N} e_{y,z,m,n}$; $y = 1,...,Y; z = 1,...,Y; m = 1,...,M.$

Thus $e_{y,z,m}$ is equal to total sales of products sold in month m of year y provided the products are also sold in month m of year z. Using definitions (33) and (34), the *expenditure share for product n in month m of year y, conditional on products being present in years y and z*, $e_{y,z,m,n}$, as follows:

(35) $s_{y,z,m,n} \equiv e_{y,z,m,n}/e_{y,z,m}$; $y = 1,...,Y; z = 1,...,Y; m = 1,...,M; n = 1,...,N.$

[33] When defining $e_{y,z,m,n}$ in a statistical programming package, it is useful to define the dummy variables, $\delta_{z,y,n} = \{1$ if $q_{z,y,n} > 0$; $\delta_{z,y,n} = 0$ if $q_{z,y,n} = 0\}$ and then define $e_{y,z,m,n}$ as $p_{y,m,n}q_{y,m,n}\delta_{z,y,n}$.

Note that if y = z, then the conditional shares $s_{y,y,m,n}$ defined by (35) collapse down to the actual expenditure shares on commodity n in month m of year y, $s_{y,m,n}$, defined by (23) above; i.e., we have:

(36) $s_{y,y,m,n} = s_{y,m,n} \equiv p_{y,m,n}q_{y,m,n}/\sum_{k=1}^{N} p_{y,m,k}q_{y,m,k}$; $\quad$ y = 1,...,Y; m = 1,...,M; n = 1,...,N.

The bilateral *maximum overlap Törnqvist Theil index that compares the prices of month m in year y to the prices of month m in year z*, $P_T^{m*}(y/z)$, is defined as follows:

(37) $P_T^{m*}(y/z) \equiv \exp[\sum_{n\in S(y,m)\cap S(z,m)} (½)(s_{y,z,m,n}+ s_{z,y,m,n})\ln(p_{y,m,n}/p_{z,m,n})]$;
y = 1,...,Y; z = 1,...,Y; m = 1,...,M.

Thus only the product prices that are positive in month m of year y and in month m of year z appear in the summations on the right hand side of definitions (37). $P_T^{m*}(y/z)$ compares the prices of month m products that are available *in both year y and year z*. The bilateral indexes $P_T^{m*}(y/z)$ defined by (37) can be used to construct the maximum overlap fixed base and chained Törnqvist Theil indexes.

The sequence of maximum overlap year over year fixed base Törnqvist Theil indexes for month m will be denoted by $P_{TFB}^{y,m*}$ for y = 1,2,...,Y. For our empirical example, Y = 6 and the *year over year maximum overlap fixed base Törnqvist Theil indexes* $P_{TFB}^{y,m*}$ for months m = 1,...,12 are defined to be the indexes $P_T^{m*}(1/1)$, $P_T^{m*}(2/1)$, $P_T^{m*}(3/1)$, $P_T^{m*}(4/1)$, $P_T^{m*}(5/1)$, $P_T^{m*}(6/1)$ where the maximum overlap link indexes $P_T^{m*}(y/z)$ are defined by (37).

Define the *year over year maximum overlap chained Törnqvist Theil index* for month m in year 1, $P_{TCH}^{1,m*}$, as unity:

(38) $P_{TCH}^{1,m*} \equiv 1$; $\quad$ m = 1,....,M.

For years following year 1, the *maximum overlap chained Törnqvist Theil indexes* for the month m in the years y = 2,...,Y, $P_{TCH}^{y,m*}$, are defined by cumulating the corresponding successive annual year over year links for month m defined by (37); i.e., we have the following definitions:

(39) $P_{TCH}^{y,m*} \equiv P_{TCH}^{y-1,m*}\, P_T^{m*}(y/(y-1))$; $\quad$ m = 1,...,M; y = 2,...,Y.

The fixed base and chained maximum overlap Törnqvist Theil indexes for our May data are listed below in Table 6.

In order to define the year over year predicted share similarity linked indexes for a particular month, we need to define the relative price dissimilarity matrix for each month. It turns out that we can still use definitions (21) and (22) to define the new dissimilarity matrix using our "new" data that does not use carry forward prices. For convenience, we repeat these definitions. Thus define the *predicted share for product n in month m of year y*, $s_{z,y,m,n}$, that uses the month m quantities of year y and the prices of month m in year z as follows:

(40) $s_{z,y,m,n} \equiv p_{z,m,n}q_{y,m,n}/p^{z,m}\cdot q^{y,m}$; $\quad$ y = 1,...,Y; z = 1,...,Y; m = 1,2,...,M; n = 1,2,...,N.

Define the *Predicted Share measure of relative price dissimilarity* between the prices of month m in year y and the prices of month m in year z, $\Delta_{PS}(p^{z,m},p^{y,m},q^{z,m},q^{y,m})$, as follows:

$$(41)\ \Delta_{PS}(p^{z,m},p^{y,m},q^{z,m},q^{y,m}) \equiv \Sigma_{n=1}^{N}\,[s_{y.m,n} - s_{z,y,m,n}]^2 + \Sigma_{n=1}^{N}\,[s_{z.m,n} - s_{y,z,m,n}]^2$$
$$= \Sigma_{n=1}^{N}\,[(p_{y,m,n}q_{y,m,n}/p^{y,m}\cdot q^{y,m}) - (p_{z,m,n}q_{y,m,n}/p^{z,m}\cdot q^{y,m})]^2$$
$$+ \Sigma_{n=1}^{N}\,[(p_{z,m,n}q_{z,m,n}/p^{z,m}\cdot q^{z,m}) - (p_{y,m,n}q_{z,m,n}/p^{y,m}\cdot q^{z,m})]^2.$$

If the products that were purchased in month m of years y and z were identical, then the "new" measure of relative price dissimilarity defined by (41) will be identical to the "old" measure defined by (22). However, in the case where prices in years y and z are not matched, (41) will generate a larger measure of price dissimilarity than was generated by the corresponding (22) measure; i.e., there is now a *penalty for a lack of price matching* (which can be large if the difference between $s_{y.m,n}$ and $s_{z.m,n}$ is large for an unmatched product n).

To see how the predicted share measure of relative price dissimilarity defined by (41) turned out for our Israeli data for the month of May when we do not use imputed prices, see Table 6 below. The month m is equal to 5 (May). As usual, the years y and z range from 1 to 6.

Table 6: May Predicted Share Measures of Price Dissimilarity Excluding Imputed Prices

m = 5	y = 1	y = 2	y = 3	y = 4	y = 5	y = 6
z = 1	0.00000	0.02471	0.02505	0.04297	0.03604	0.03227
z = 2	0.02471	0.00000	0.00988	0.02858	0.01565	0.01926
z = 3	0.02505	0.00988	0.00000	0.01226	0.00409	0.01042
z = 4	0.04297	0.02858	0.01226	0.00000	0.01060	0.00204
z = 5	0.03604	0.01565	0.00409	0.01060	0.00000	0.01445
z = 6	0.03227	0.01926	0.01042	0.00204	0.01445	0.00000

The predicted share measures of relative price dissimilarity listed in Table 6 have a mean equal to 0.01601 whereas the measures for Table 3 in the previous section had a mean equal to 0.0089. Thus *excluding* the use of imputed prices for the predicted share measures of dissimilarity for our May year over year data substantially *increased* the resulting measures of price dissimilarity. The predicted share measures of price dissimilarity grow in magnitude when imputed prices are replaced by zero prices because the measures impose a substantial penalty for a lack of price matching.[34]

The new real time predicted share relative price similarity linked price indexes for May (that exclude the use of imputed prices), $P_S^{y,5*}$, are constructed as follows. Set $P_S^{1,5*} \equiv 1$. The year over year index for May in year 2 is set equal to the maximum overlap bilateral Fisher index $P_F^{m*}(y/z)$ where m = 5, y = 2 and z = 1 (see definition (26) above). Thus the year 2 similarity linked index for May is $P_S^{2,5*} \equiv P_F^{5*}(2/1)$. Now look down the y = 3 column in Table 6. We need to link year 3 to either year 1 or year 2. The dissimilarity measures for these two years relative to year 3 are 0.02505 and 0.00988 respectively. The degree of relative price dissimilarity is far smaller for the link to year 2 than it is to year 1 so we use the maximum overlap Fisher link (for the month 5 data) from period 2 to period 3, $P_F^{5*}(3/2)$, to construct the year 3 similarity linked index for May as $P_S^{3,5*} \equiv P_S^{2,5*}\times P_F^{5*}(3/2)$. Now we need to link year 4 to either year 1, 2 or 3. Look down the y = 4 column in Table 6 to find the lowest dissimilarity measure above the main diagonal of the matrix. The smallest of the 3 numbers 0.04297, 0.02858 and 0.01226 is 0.01226. Thus we link the year 4 May data to the year 3 May data using the maximum overlap Fisher May link from year 3 to year 4, $P_F^{5*}(4/3)$, and the year 4 similarity linked index value is $P_S^{4,5*} \equiv P_S^{3,5*}\times P_F^{5*}(4/3)$. Thus each year, as the new May data become available, we use the maximum overlap Fisher bilateral index that links the new period to the previous period that has the lowest measure of relative price

[34] See the discussion of the predicted share multilateral method in Diewert (2021b).

dissimilarity. The final two bilateral links are year 5 to year 3 and year 6 to year 4. The resulting year 5 and 6 similarity linked index values are $P_S^{5,5*} \equiv P_S^{3,5*} \times P_F^{5*}(5/3)$ and $P_S^{6,5*} \equiv P_S^{4,5*} \times P_F^{5}(6/4)$. The set of optimal real time bilateral links for the May data can be summarized as follows:

```
1 – 2 – 3 – 4
        |   |
        5   6.
```

The new set of May bilateral links is different from the set of bilateral links for May that used carry forward and carry backward prices. To see the differences between the carry forward indexes for May listed in Table 4 in the previous section with the corresponding maximum overlap indexes for May that are described above, see Table 7 below. The indexes listed in Table 7 do not use any imputed prices in their construction.

Table 7: Year over Year Maximum Overlap Indexes for May

Year y	$P_{LFB}^{y,5*}$	$P_{PFB}^{y,5*}$	$P_{FFB}^{y,5*}$	$P_{TFB}^{y,5*}$	$P_{LCH}^{y,5*}$	$P_{PCH}^{y,5*}$	$P_{FCH}^{y,5*}$	$P_{TCH}^{y,5*}$	$P_{GEKS}^{y,5*}$	$P_S^{y,5*}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.95007	0.91814	0.93397	0.93252	0.95007	0.91814	0.93397	0.93252	0.94462	0.93397
3	1.05674	1.03102	1.04380	1.04354	1.06935	0.99802	1.03307	1.03104	1.05052	1.03307
4	1.33870	1.26554	1.30161	1.29967	1.33429	1.21827	1.27496	1.27191	1.29677	1.27496
5	1.17963	1.17093	1.17527	1.17658	1.21701	1.06707	1.13958	1.13863	1.16610	1.13587
6	1.34224	1.29900	1.32044	1.31917	1.36461	1.18740	1.27293	1.27122	1.31228	1.28980
Mean	1.14460	1.11410	1.12920	1.12860	1.15590	1.06480	1.10910	1.10760	1.12840	1.11130

As was the case for the carry forward indexes listed in Table 4 above, the maximum overlap fixed base and chained Laspeyres indexes for May, $P_{LFB}^{y,5*}$ and $P_{LCH}^{y,5*}$, listed in Table 7 end up well above the superlative indexes and the maximum overlap fixed base and the chained Paasche indexes for May, $P_{PFB}^{y,5*}$ and $P_{PCH}^{y,5*}$, end up well below the superlative indexes. The remaining six superlative indexes (the fixed base and chained Fisher indexes, $P_{FFB}^{y,5*}$ and $P_{FCH}^{y,5*}$, the fixed base and chained Törnqvist Theil indexes, $P_{TFB}^{y,5*}$ and $P_{TCH}^{y,5*}$, the GEKS indexes $P_{GEKS}^{y,5*}$ and the predicted share similarity linked indexes $P_S^{y,5*}$) ended up in year 6 at 1.3204, 1.2729, 1.31917, 1.27122, 1.3123 and 1.2898 respectively. It appears that the chained Fisher and Törnqvist Theil indexes suffer from some downward chain drift since the other four superlative indexes are free of chain drift and they ended up (on average) about 3.77 percentage points above where the average of the two chained superlative indexes ended. Thus for our May data, it appears that *the use of carry forward prices for missing product prices led to a substantial downward bias*. Thus the use of carry forward prices to replace missing prices is not recommended.

The year over year indexes for all 12 months are reported in Table A.22 in the Appendix. The following table reports the overall mean and variance for all 8 indexes, where the index values are stacked into a single column with 72 rows for each of the 8 indexes. The averages reported in Table 8 use maximum overlap indexes whereas the corresponding averages reported in Table 5 used carry forward prices which will tend to give lower indexes given that there was general fruit inflation in Israel for the 6 years in our sample. The averages reported in Table 8 are in fact higher than the corresponding averages in Table 5 with the exceptions of the chained Paasche and chained Fisher indexes.

Table 8: Year over Year Index Means and Variances Over All Months and Years for Ten

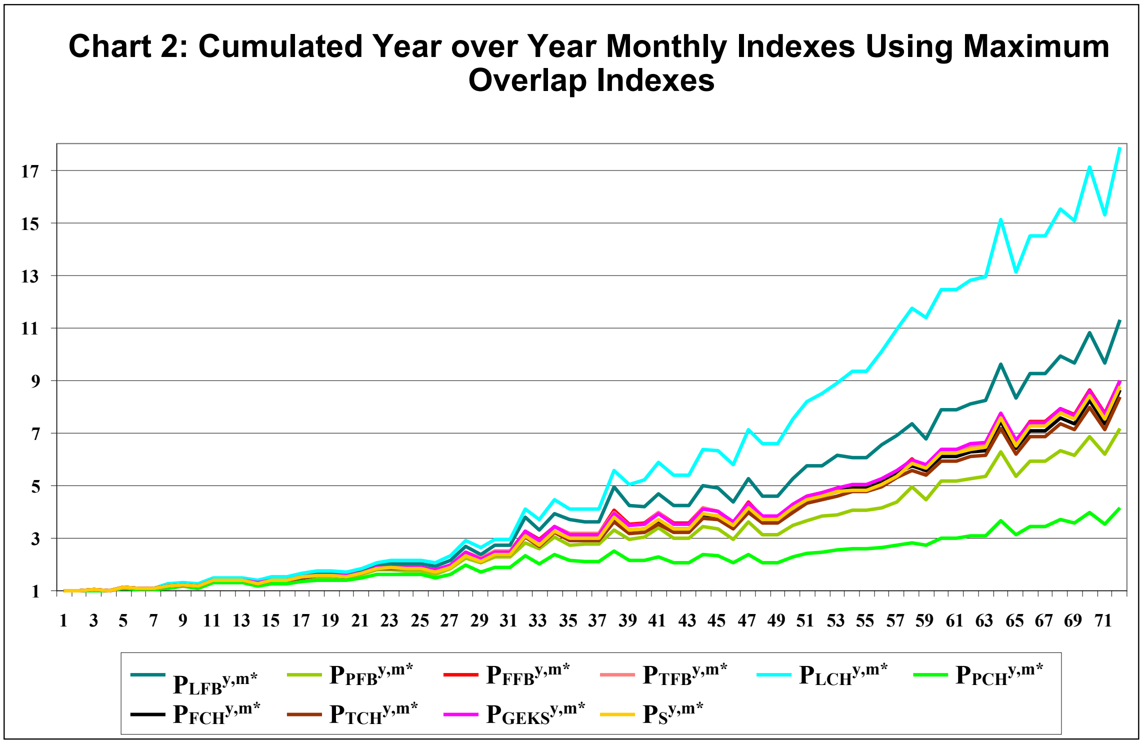
Indexes Using Maximum Overlap Bilateral Indexes

	$P_{LFB}^{y,m*}$	$P_{PFB}^{y,m*}$	$P_{FFB}^{y,m*}$	$P_{TFB}^{y,m*}$	$P_{LCH}^{y,m*}$	$P_{PCH}^{y,m*}$	$P_{FCH}^{y,m*}$	$P_{TCH}^{y,m*}$	$P_{GEKS}^{y,m*}$	$P_S^{y,m*}$
Mean	**1.1381**	**1.1003**	**1.1189**	**1.1177**	**1.1591**	**1.0765**	**1.1163**	**1.1136**	**1.1187**	**1.1184**
Variance	**0.0167**	**0.0101**	**0.0128**	**0.0126**	**0.0209**	**0.0076**	**0.0119**	**0.0115**	**0.0125**	**0.0123**

As usual, the fixed base and chained Laspeyres maximum overlap indexes, $P_{LFB}^{y,m*}$ and $P_{LCH}^{y,m*}$, and the fixed base and chained Paasche maximum overlap indexes, $P_{PFB}^{y,m*}$ and $P_{PCH}^{y,m*}$, have some considerable amounts of upward and downward substitution bias relative to the remaining superlative indexes. The chained Fisher and chained Törnqvist Theil indexes, $P_{FFB}^{y,5*}$ and $P_{TCH}^{y,5*}$, appear to have some amount of downward chain drift bias relative to the remaining four superlative indexes, which are free of chain drift bias by construction. The fixed base Fisher, fixed base Törnqvist Theil, GEKS and Similarity Linked indexes, $P_{FFB}^{y,m*}$, $P_{TFB}^{y,5*}$, $P_{GEKS}^{y,m*}$ and $P_S^{y,m*}$, all have about the same mean and variance and appear to be equally satisfactory for our particular empirical example. The means for these four maximum overlap indexes over all months was 1.1189, 1.1177, 1.1187 and 1.1184 and the average of these four averages is 1.1184. The corresponding means for the carry forward indexes $P_{FFB}^{y,m}$, $P_{TFB}^{y,m}$, $P_{GEKS}^{y,m}$ and $P_S^{y,m}$ from Table 5 are 1.1180, 1.1170, 1.1111 and 1.1178 and the average of these four averages is 1.1160. Thus the use of carry forward prices leads to an average downward bias of about 0.24 percentage points compared to the corresponding maximum overlap indexes for our best index number formulae for our particular empirical example. This is a significant downward bias.[35]

In order to illustrate the differences between the ten different index number formulae, we cumulated the ten year over year indexes listed in Table A.22 in the Appendix and plotted the resulting cumulated indexes on Chart 2 below. The construction of the cumulated series for each index formula follows the same as the process we used to construct Chart 1.

[35] One sixth of the indexes listed in Tables A.21 and A.22 are equal to one so the actual bias is even larger.



The indexes plotted on Chart 2 are very close to their counterparts on Chart 1. For the most part, the indexes on Chart 2 are a bit above their counterparts on Chart 1 due to the fact that the Chart 1 indexes used carry forward prices, which tend to lower measured inflation in a period of general inflation. The highest series on Chart 2 is the cumulated chained Laspeyres index P_{LCH}^{*} followed by the cumulated fixed base Laspeyres index, P_{LFB}^{*}. The lowest series is the cumulated chained Paasche index P_{PCH}^{*} followed by the cumulated fixed base Paasche index, P_{PFB}^{*}. The remaining 6 indexes are all clustered together in the middle of these outlier series.

Our conclusions regarding the use of year over year monthly indexes at this point are as follows:

- The use of the Laspeyres and Paasche indexes should be avoided. The fixed base and chained Laspeyres indexes tend to lie well above the clustered superlative indexes while the fixed base and chained Paasche indexes tend to lie well below the clustered superlative indexes.
- The chained Fisher and Törnqvist Theil indexes may suffer from a small amount of chain drift.
- The fixed base Fisher, Törnqvist Theil, GEKS and Predicted Share Similarity linked indexes are all fairly close to each other in the present context where we are measuring year over year inflation for each month in the year.
- The use of carry forward prices will tend to lead to indexes which are biased downward if there is general inflation and so in order to avoid this potential bias, it is best to use the indexes that use maximum overlap superlative bilateral indexes as their basic building blocks. Thus the maximum overlap fixed base Fisher and fixed base Törnqvist Theil, the GEKS and the Predicted Share similarity linked indexes, $P_{FFB}^{y,m*}$, $P_{TFB}^{y,m*}$, $P_{GEKS}^{y,m*}$ and $P_{S}^{y,m*}$, emerge as our "best" choices for year over year monthly indexes.

In the following section, we turn our attention to annual price indexes.

4. The Construction of Annual Indexes using Carry Forward Prices

Assuming that each commodity in each season of the year is a separate "annual" commodity is the simplest and theoretically most satisfactory method for dealing with seasonal commodities when the goal is to construct annual price and quantity indexes. This idea can be traced back to Mudgett in the consumer price context and to Stone in the producer price context:

"The basic index is a yearly index and as a price or quantity index is of the same sort as those about which books and pamphlets have been written in quantity over the years." Bruce D. Mudgett (1955; 97).

"The existence of a regular seasonal pattern in prices which more or less repeats itself year after year suggests very strongly that the varieties of a commodity available at different seasons cannot be transformed into one another without cost and that, accordingly, in all cases where seasonal variations in price are significant, the varieties available at different times of the year should be treated, in principle, as separate commodities." Richard Stone (1956; 74-75).

Using carry forward prices for missing products and using the notation explained in section 2 above, the N dimensional price and quantity vectors for month m in year y are defined as $p^{y,m} \equiv [p_{y,m,1}, p_{y,m,2},...,p_{y,m,N}]$ and $q^{y,m} \equiv [q_{y,m,1},q_{y,m,2},...,q_{y,m,N}]$ for y = 1,...,Y and m = 1,...,M.[36] The *year y annual price and quantity vectors* are defined as the NM dimensional vectors $p^y \equiv [p^{y,1}, p^{y,2},..., p^{y,M}]$ and $q^y \equiv [q^{y,1}, q^{y,2},..., q^{y,M}]$ respectively for y = 1,...,Y. Using this new notation, the year y *annual fixed base Laspeyres price index* using carry forward prices is defined as follows:

$$\begin{aligned}(42)\ P_{LFB}^y &\equiv p^y\cdot q^1/p^1\cdot q^1\ ; \qquad\qquad y = 1,...,Y;\\ &= \Sigma_{m=1}^M\, p^{y,m}\cdot q^{1,m}/\Sigma_{m=1}^M\, p^{1,m}\cdot q^{1,m}\\ &= \Sigma_{m=1}^M\, [p^{y,m}\cdot q^{1,m}/p^{1,m}\cdot q^{1,m}][p^{1,m}\cdot q^{1,m}/\Sigma_{m=1}^M\, p^{1,m}\cdot q^{1,m}]\\ &= \Sigma_{m=1}^M\, S_{1,m}\, P_{LFB}^{y,m}\end{aligned}$$

where $S_{1,m} \equiv p^{1,m}\cdot q^{1,m}/\Sigma_{m=1}^M\, p^{1,m}\cdot q^{1,m}$ is the month m share of total year 1 expenditure on the seasonal commodities in scope and $P_{LFB}^{y,m} \equiv p^{y,m}\cdot q^{1,m}/p^{1,m}\cdot q^{1,m}$ is the Laspeyres fixed base price index for month m in year y which was defined by (2) in section 2.[37] Thus *the annual fixed base Laspeyres price index for year y*, P_{LFB}^y, is a year 1 monthly expenditure share weighted *arithmetic* average of the M year over year fixed base Laspeyres monthly indexes for year y. These annual fixed base Laspeyres indexes are listed in Table 10 below for our Israeli data set.

The year y *annual fixed base Paasche index* using carry forward prices is defined as follows:

$$\begin{aligned}(43)\ P_{PFB}^y &\equiv p^y\cdot q^y/p^1\cdot q^y\ ; \qquad\qquad y = 1,...,Y;\\ &= 1/[p^1\cdot q^y/p^y\cdot q^y]\\ &= [\Sigma_{m=1}^M\, p^{1,m}\cdot q^{y,m}//\Sigma_{m=1}^M\, p^{y,m}\cdot q^{y,m}]^{-1}\end{aligned}$$

[36] The quantity $q_{y,m,n}$ is the quantity of product n purchased in month m of year y; if no amount of this product was purchased in month m of year y, $q_{y,m,n} = 0$. If product n was never purchased in any month, $p_{y,m,n} = 0$. If some amount of product n was purchased in month m of any year y = 1,...,Y, then $p_{y,m,n}$ is the actual unit value price if product n was purchased in year y; otherwise $p_{y,m,n}$ is a carry forward or carry backward price. The share of product n in the monthly expenditure on all products in month m of year y is defined as $s_{y,m,n} \equiv p_{y,m,n}q_{y,m,n}/\Sigma_{k\in S(m)}\, p_{y,m,k}q_{y,m,k} = p_{y,m,n}q_{y,m,n}\,/p^{y,m}\cdot q^{y,m}$ for y = 1,...,Y; m = 1,2,...,M ; n = 1,...,N.

[37] The new definition for $P_{LFB}^{y,m}$ is equivalent to definition (2).

$= [(\Sigma_{m=1}^{M} p^{1,m}\cdot q^{y,m}/p^{y,m}\cdot q^{y,m})/(p^{y,m}\cdot q^{y,m}/\Sigma_{m=1}^{M} p^{y,m}\cdot q^{y,m})]^{-1}$
$= [\Sigma_{m=1}^{M} S_{y,m} (P_{PFB}^{y,m})^{-1}]^{-1}$

where $P_{PFB}^{y,m} \equiv p^{y,m}\cdot q^{y,m}/p^{1,m}\cdot q^{y,m}$ is the Paasche fixed base price index for month m in year y which was defined by (3) in section 2 [38] and the *month m shares of annual expenditures on the seasonal commodities in scope for year y*, $S_{y,m}$, is defined as follows:

(44) $S_{y,m} \equiv p^{y,m}\cdot q^{y,m}/\Sigma_{k=1}^{M} p^{y,k}\cdot q^{y,k}$; $\quad m = 1,...,M;\ y = 1,...,Y.$

Thus *the annual fixed base Paasche price index for year y*, P_{PFB}^{y}, is a year y monthly expenditure share weighted *harmonic* average of the M fixed base year over year Paasche monthly indexes for year y. These annual fixed base Paasche indexes are listed in Table 10 below for our Israeli data set.

The year y *annual fixed base Fisher index* is defined as the geometric mean of the annual Laspeyres and Paasche indexes defined by (42) and (43):

(45) $P_{FFB}^{y} = [P_{LFB}^{y}\, P_{PFB}^{y}]^{1/2}$; $\quad y = 1,...,Y.$

In section 2, recall that the fixed base Törnqvist Theil indexes for month m in year y were defined as $P_{TFB}^{y,m} \equiv \exp[\Sigma_{n\in S(m)} (½)(s_{1,m,n}+ s_{y,m,n})\ln(p_{y,m,n}/p_{1,m,n})]$ for m = 1,...,12; y = 1,...,Y. The *fixed base annual Törnqvist-Theil index* for year y using carry forward prices is defined as follows:

(46) $P_{TFB}^{y} \equiv \exp[\Sigma_{m=1}^{M} \Sigma_{n\in S(m)} (½)(S_{1,m}s_{1,m,n} + S_{y,m}s_{y,m,n})\ln(p_{y,m,n}/p_{1,m,n})]$; $\quad y = 1,...,Y$

where the within month expenditure shares $s_{y,m,n}$ are defined by (1) and the month m expenditure shares in year y, $S_{y,m}$, are defined by (44).

In order to define the annual chained Laspeyres, Paasche, Fisher and Törnqvist Theil indexes as well as the annual GEKS indexes, it is necessary to define bilateral annual Laspeyres, Paasche, Fisher and Törnqvist Theil indexes for all pairs of years y and z. Thus define these *bilateral annual indexes* that compare the prices of year y relative to the base year z as follows:

(47) $P_L(y/z) \equiv p^y\cdot q^z/p^z\cdot q^z$; $\quad z = 1,...,Y;\ y = 1,...,Y;$
(48) $P_P(y/z) \equiv p^y\cdot q^y/p^z\cdot q^y$; $\quad z = 1,...,Y;\ y = 1,...,Y;$
(49) $P_F(y/z) \equiv [P_L(y/z)P_P(y/z)]^{1/2}$; $\quad z = 1,...,Y;\ y = 1,...,Y;$
(50) $P_T(y/z) \equiv \exp[\Sigma_{m=1}^{M} \Sigma_{n\in S(m)} (½)(S_{z,m}s_{z,m,n} + S_{y,m}s_{y,m,n})\ln(p_{y,m,n}/p_{z,m,n})]$; $\quad z = 1,...,Y;\ y = 1,...,Y.$

The annual chained Laspeyres, Paasche, Fisher and Törnqvist-Theil indexes are defined as follows for year 1:

(51) $P_{LCH}^{1} \equiv 1$; $P_{PCH}^{1} \equiv 1$; $P_{FCH}^{1} \equiv 1$; $P_{TCH}^{1} \equiv 1$.

For years y following year 1, the above annual chained indexes are defined recursively using the annual bilateral indexes defined above by (47)-(50) as follows:

(52) $P_{LCH}^{y} \equiv P_{LCH}^{y-1}\, P_L(y/(y-1))$; $\quad y = 2,...,Y;$
(53) $P_{PCH}^{y} \equiv P_{PCH}^{y-1}\, P_P(y/(y-1))$; $\quad y = 2,...,Y;$

[38] The new definition for $P_{PFB}^{y,m}$ is equivalent to definition (3).

(54) $P_{FCH}^{y} \equiv P_{FCH}^{y-1}\, P_F(y/(y-1))$; $\quad y = 2,...,Y;$
(55) $P_{TCH}^{y} \equiv P_{TCH}^{y-1}\, P_T(y/(y-1))$; $\quad y = 2,...,Y.$

The Fisher fixed base index for year y, P_{FFB}^{y}, defined above by (45) chose year 1 as the base period and formed the following sequence of year over year price levels relative to year 1: $P_F(1/1) = 1$, $P_F(2/1)$, $P_F(3/1)$, ..., $P_F(Y/1)$. But one could also use year 2 as the base period and use the following sequence of price levels to measure annual inflation for each year y: $P_F(1/2)$, $P_F(2/2) = 1$, $P_F(3/2)$, ..., $P_F(Y/2)$. Each year could be chosen as the base period and thus we end up with Y alternative series of Fisher price levels for each year. Since each of these sequences of price levels is equally plausible, following Gini (1924) (1931), Eltetö and Köves (1964) and Szulc (1964), the *GEKS price levels*, P_{GEKS}^{y}, for years y = 1,2,...,Y are defined as the geometric mean of the separate indexes we obtain by using each year as the base year:

(56) $P_{GEKS}^{y} \equiv [\prod_{z=1}^{Y} P_F(y/z)]^{1/Y}$; $\quad y = 1,...,Y.$

Note that each choice of a base year z is treated in a symmetric manner in the above definitions. The annual *GEKS price indexes* P_{GEKS}^{y} are obtained by normalizing the above price levels so that the year 1 index is equal to 1. Thus we have the following definitions for the annual GEKS index for year y (using carry forward prices), P_{GEKS}^{y}:

(57) $P_{GEKS}^{y} \equiv p_{GEKS}^{y}/p_{GEKS}^{1}$; $\quad y = 1,...,Y.$

The annual GEKS price indexes using carry forward prices are also listed in Table 10 below using the data from our empirical example.

The basic building blocks used to form the GEKS multilateral index are the bilateral Fisher indexes $P_F(y/z)$. It is not necessary to use the Fisher bilateral indexes as the basic building blocks; instead the bilateral Törnqvist Theil indexes $P_T(y/z)$ defined by (50) could be used.[39] Thus following Caves, Christensen and Diewert (1982) and Inklaar and Diewert (2016), the *CCDI price levels*, p_{CCDI}^{y}, for years y = 1,2,...,Y are defined as the geometric mean of the separate indexes we obtain by using each year as the base year using the $P_T(y/z)$ as the bilateral building blocks :

(58) $p_{CCDI}^{y} \equiv [\prod_{z=1}^{Y} P_T(y/z)]^{1/Y}$; $\quad y = 1,...,Y.$

The annual CCDI index for year y, P_{CCDI}^{y}, is defined as the following normalization of the CCDI price levels:

(59) $P_{CCDI}^{y} \equiv p_{CCDI}^{y}/p_{CCID}^{1}$; $\quad y = 1,...,Y.$

The 10 annual indexes P_{LFB}^{y}, P_{PFB}^{y}, P_{FFB}^{y}, P_{TFB}^{y}, P_{LCH}^{y}, P_{PCH}^{y}, P_{FCH}^{y}, P_{TCH}^{y}, P_{GEKS}^{y} and P_{CCDI}^{y} that use year over year carry forward prices for our empirical example are listed in Table 10 below.

Our final annual Mudgett Stone annual index that uses year over year carry forward prices for missing prices is the Predicted Share Similarity linked index P_S^{y}.

[39] Caves, Christensen and Diewert (1982) defined the quantity index counterpart to the price index defined by (58) using a different representation of the index. Inklaar and Diewert (2016) showed that the CCD definition was equivalent to the index defined by (58) definition. Thus the multilateral indexes defined by (58) are called the CCDI indexes. They are also called GEKS Törnqvist indexes by statistical agencies.

The year y, month m, product n actual expenditure share is $s_{y,m,n} \equiv p_{y,m,n}q_{y,m,n}/p^{y,m}\cdot q^{y,m}$. The prediction for this share using the price of product n of month m in year z, $p_{z,m,n}$, and the actual quantity of product n for month m in year y is the *predicted share* $s_{z,y,m,n} \equiv p_{z,m,n}q_{y,m,n}/p^{z,m}\cdot q^{y,m}$ for $n = 1,...,N$, $m = 1,...,M$, $z = 1,...,Y$ and $y = 1,...,Y$. The new annual measure of *Predicted Share Price Dissimilarity* between the prices of years z and y, $\Delta_{PSA}(p^z,p^y,q^z,q^y)$, is defined as follows:

$$(60)\ \Delta_{PSA}(p^z,p^y,q^z,q^y) \equiv \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} [s_{y.m,n} - s_{z,y,m,n}]^2 + \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} [s_{z.m,n} - s_{y,z,m,n}]^2$$
$$= \Sigma_{m=1}^{M} \Delta_{PS}(p^{z,m},p^{y,m},q^{z,m},q^{y,m})$$

where $\Delta_{PS}(p^{z,m},p^{y,m},q^{z,m},q^{y,m}) \equiv \Sigma_{n=1}^{N} [s_{y.m,n} - s_{z,y,m,n}]^2 + \Sigma_{n=1}^{N} [s_{z.m,n} - s_{y,z,m,n}]^2$ is the month m measure of *monthly* price dissimilarity between the product prices of month m in years z and y that was defined in section 2 by definitions (22). Thus the new annual measure of price dissimilarity (using carry forward prices) is equal to the sum over the M monthly product price dissimilarity measures for month m prices in years z and y using carry forward prices.

Here is the table of the bilateral measures of Annual Predicted Share Price Dissimilarity for our empirical example.

Table 9: Annual Predicted Share Measures of Price Dissimilarity Using Carry Forward Prices

	y = 1	y = 2	y = 3	y = 4	y = 5	y = 6
z = 1	**0.00000**	**0.00196**	**0.00198**	**0.00176**	**0.00173**	**0.00225**
z = 2	**0.00196**	**0.00000**	**0.00107**	**0.00207**	**0.00109**	**0.00264**
z = 3	**0.00198**	**0.00107**	**0.00000**	**0.00104**	**0.00068**	**0.00099**
z = 4	**0.00176**	**0.00207**	**0.00104**	**0.00000**	**0.00129**	**0.00055**
z = 5	**0.00173**	**0.00109**	**0.00068**	**0.00129**	**0.00000**	**0.00102**
z = 6	**0.00225**	**0.00264**	**0.00099**	**0.00055**	**0.00102**	**0.00000**

The *real time* set of bilateral links which minimize the predicted share measures of relative price dissimilarity for the annual data are as follows: link 2 to 1; 3 to 2; 4 to 3; 5 to 3 and 6 to 4. The optimal set of bilateral links can be summarized as follows:

1 – 2 – 3 – 4
          | |
          5   6

Thus we define $P_S^1 \equiv 1$, $P_S^2 \equiv P_F(2/1)$, $P_S^3 \equiv P_S^2 \times P_F(3/2)$, $P_S^4 \equiv P_S^3 \times P_F(4/3)$, $P_S^5 \equiv P_S^3 \times P_F(5/3)$ and $P_S^6 \equiv P_S^4 \times P_F(6/4)$ where the bilateral annual Fisher indexes $P_F(y/z)$ are defined by definitions (49).

The 11 annual indexes P_{LFB}^y, P_{PFB}^y, P_{FFB}^y, P_{TFB}^y, P_{LCH}^y, P_{PCH}^y, P_{FCH}^y, P_{TCH}^y, P_{GEKS}^y, P_{CCDI}^y and P_S^y that use year over year carry forward prices for our empirical example are listed in Table 10 below and plotted in Chart 3 below.

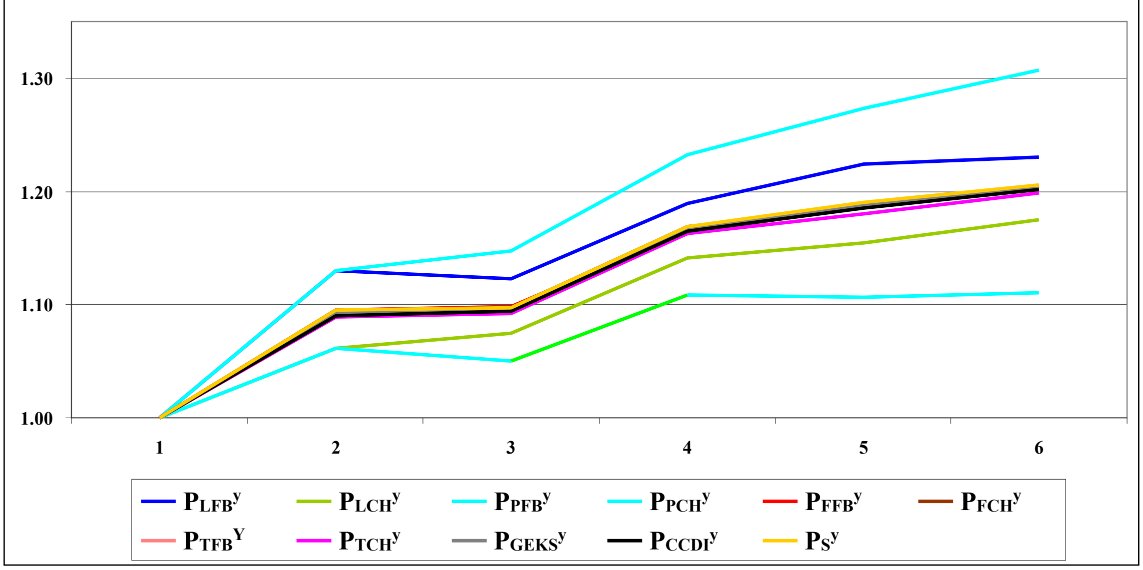


Chart 3: Annual Indexes using Year over Year Carry Forward Prices

Table 10: Alternative Annual Mudgett Stone Indexes that Use Year over Year Carry Forward Prices

Year	P_{LFB}^y	P_{PFB}^y	P_{LCH}^y	P_{PCH}^y	P_{FFB}^y	P_{FCH}^y	P_{TFB}^y	P_{TCH}^y	P_{GEKS}^y	P_{CCDI}^y	P_S^y
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.1299	1.0611	1.1299	1.0611	1.0950	1.0950	1.0892	1.0892	1.0929	1.0900	1.0950
3	1.1224	1.0745	1.1470	1.0502	1.0982	1.0975	1.0963	1.0918	1.0966	1.0941	1.0975
4	1.1891	1.1411	1.2322	1.1083	1.1648	1.1686	1.1624	1.1626	1.1676	1.1647	1.1686
5	1.2241	1.1549	1.2729	1.1060	1.1890	1.1865	1.1871	1.1805	1.1884	1.1856	1.1903
6	1.2306	1.1752	1.3070	1.1102	1.2026	1.2046	1.2015	1.1988	1.2044	1.2020	1.2056
Mean	1.1494	1.1011	1.1815	1.0726	1.1249	1.1254	1.1227	1.1205	1.1250	1.1227	1.1262

It can be seen that the annual fixed base and chained Laspeyres indexes, P_{LFB}^y and P_{LCH}^y, lie well above the superlative indexes and the annual fixed base and chained Paasche indexes, P_{PFB}^y and P_{PCH}^y, lie well below the remaining indexes. The remaining indexes are all tightly clustered together and cannot be easily distinguished on a chart. Chart 3 below plots the 11 indexes listed in Table 10 above.

Thus for our particular empirical example, all of the annual indexes that are exact for a flexible functional form give much the same answer when we use year over year carry forward prices. However, looking at the averages listed in Table 10, it can be seen that the three indexes that use bilateral Törnqvist Theil indexes as building blocks, P_{TFB}^y, P_{TCH}^y and P_{CCDI}^y, have slightly lower average index values compared to the indexes that use bilateral Fisher indexes as building blocks. In section 3, we saw that the use of year over year carry forward prices for missing prices tended to lead to indexes which were lower than the counterpart indexes that did not use any imputed prices. We will see if the same tendency occurs when we compute annual Mudgett Stone indexes using annual bilateral maximum overlap indexes.

5. The Construction of Annual Indexes using Maximum Overlap Bilateral Indexes

In order to define the annual Laspeyres, Paasche, Fisher and Törnqvist Theil indexes without using imputations for missing prices, it is necessary to define imputation free bilateral annual Laspeyres, Paasche, Fisher and Törnqvist Theil indexes for all pairs of years y and z. Thus define the following *maximum overlap bilateral annual indexes* that compare the prices of year y relative to the base year z for products n that were available in years y and z as follows for z = 1,...,Y; y = 1,...,Y:

(61) $P_L^*(y/z) \equiv \Sigma_{m=1}^M \Sigma_{n\in S(y,m)\cap S(z,m)} p_{y,m,n}q_{z,m,n}/\Sigma_{m=1}^M \Sigma_{n\in S(y,m)\cap S(z,m)} p_{z,m,n}q_{z,m,n}$;
(62) $P_P^*(y/z) \equiv \Sigma_{m=1}^M \Sigma_{n\in S(y,m)\cap S(z,m)} p_{y,m,n}q_{y,m,n}/\Sigma_{m=1}^M \Sigma_{n\in S(y,m)\cap S(z,m)} p_{z,m,n}q_{y,m,n}$;
(63) $P_F^*(y/z) \equiv [P_L^*(y/z)P_P^*(y/z)]^{1/2}$;
(64) $P_T^*(y/z) \equiv \exp[\Sigma_{m=1}^M \Sigma_{n\in S(y,m)\cap S(z,m)} (½)(\sigma_{y,z,,m,n} + \sigma_{z,y,m,n})\ln(p_{y,m,n}/p_{z,m,n})]$

where $S(y,m)\cap S(z,m)$ is the set of products n that are available in *both* years y and z for month m. The price of product n in month m of year y, $p_{y,m,n}$, is the unit value price for that product if it is purchased in month m of year y and it is set equal to 0 if the product is not available or not sold.[40] The corresponding quantity, $q_{y,m,n}$, is the actual quantity of product n that is sold in month m of year y (which will equal 0 if the product is not available or not sold). Thus carry forward prices are not used in definitions (61)-(64). The *conditional expenditure shares*, $\sigma_{y,z,m,n}$, which appear in definitions (64) need some explanation, which is provided below.

The *actual expenditure on product n in month m of year y* is equal to $e_{y,m,n}$ defined as follows;

(65) $e_{y,m,n} \equiv p_{y,m,n}q_{y,m,n}$; $\quad$ y = 1,...,Y; m = 1,...,M; n = 1,...,N.

The *conditional on year z expenditure on product n in month m of year y*, $e_{y,z,m,n}$, is defined as actual expenditure on product n in month m of year y *if* the same product n is also sold in month m of year z and is defined to be 0 *if* product n is not sold in month m of year z. Thus the formal definition for $e_{y,z,m,n}$ is the following one:

(66) $e_{y,z,m,n} \equiv e_{y,m,n}$ if $e_{z,m,n} > 0$; $\quad$ y = 1,...,Y; z = 1,...,Y; m = 1,...,M; n = 1,...,N
$\qquad\equiv 0 \quad$ if $e_{z,m,n} = 0$.

Thus $e_{y,z,m,n}$ will be positive only if product n is purchased in month m of years y and z. *Total year y expenditures on commodities that are available in both years y and z*, $E_{y,z}$, is defined as follows:

(67) $E_{y,z} \equiv \Sigma_{m=1}^M \Sigma_{n=1}^N e_{y,z,m,n}$; $\quad$ y = 1,...,Y; z = 1,...,Y.

The *year y conditional on year z expenditure share on product n in month m of year y*, $\sigma_{y,z,m,n}$, is defined as follows:

(68) $\sigma_{y,z,m,n} \equiv e_{y,z,m,n}/E_{y,z}$; $\quad$ y = 1,...,Y; z = 1,...,Y; m = 1,...,M; n = 1,...,N.

The conditional share $\sigma_{y,z,m,n}$ is positive only if product n in month m is sold in both years y and z. These shares appear in definitions (64) above.

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[40] If a product is available in month m of year y but not purchased, we treat it as if it were an unavailable product for that month.

The *maximum overlap annual fixed base Laspeyres, Paasche, Fisher and Törnqvist Theil indexes*, $P_{LFB}{}^{y*}$, $P_{PFB}{}^{y*}$, $P_{FFB}{}^{y*}$ and $P_{TFB}{}^{y*}$, are defined as follows:

(69) $P_{LFB}{}^{y*} \equiv P_L{}^*(y/1)$; $P_{PFB}{}^{y*} \equiv P_P{}^*(y/1)$; $P_{FFB}{}^{y*} \equiv P_F{}^*(y/1)$; $P_{TFB}{}^{y*} \equiv P_T{}^*(y/1)$; $\quad y = 1,\ldots,Y.$

The *maximum overlap annual chained Laspeyres, Paasche, Fisher and Törnqvist-Theil indexes* are defined as follows for year 1:

(70) $P_{LCH}{}^{1*} \equiv 1$; $P_{PCH}{}^{1*} \equiv 1$; $P_{FCH}{}^{1*} \equiv 1$; $P_{TCH}{}^{1*} \equiv 1$.

For years y following year 1, the above indexes are defined recursively using the bilateral maximum overlap annual indexes defined above by (55)-(58) as follows:

(71) $P_{LCH}{}^{y*} \equiv P_{LCH}{}^{y-1*}\, P_L{}^*(y/(y-1))$; $\quad y = 2,\ldots,Y;$
(72) $P_{PCH}{}^{y*} \equiv P_{PCH}{}^{y-1*}\, P_P{}^*(y/(y-1))$; $\quad y = 2,\ldots,Y;$
(73) $P_{FCH}{}^{y*} \equiv P_{FCH}{}^{y-1*}\, P_F{}^*(y/(y-1))$; $\quad y = 2,\ldots,Y;$
(74) $P_{TCH}{}^{y*} \equiv P_{TCH}{}^{y-1*}\, P_T{}^*(y/(y-1))$; $\quad y = 2,\ldots,Y.$

The *maximum overlap annual GEKS price level*s, $p_{GEKS}{}^{y*}$, are defined as follows:

(75) $p_{GEKS}{}^{y*} \equiv [\prod_{z=1}{}^{Y} P_F{}^*(y/z)]^{1/Y}$; $\quad y = 1,\ldots,Y.$

The *maximum overlap annual GEKS price indexes* $P_{GEKS}{}^{y*}$ are defined as follows:

(76) $P_{GEKS}{}^{y*} \equiv p_{GEKS}{}^{y*}/p_{GEKS}{}^{1*}$; $\quad y = 1,\ldots,Y.$

The *maximum overlap CCDI price level*s, $p_{CCDI}{}^{y*}$, for year y are defined as follows

(77) $p_{CCDI}{}^{y*} \equiv [\prod_{z=1}{}^{Y} P_T{}^*(y/z)]^{1/Y}$; $\quad y = 1,\ldots,Y.$

The *maximum overlap annual CCDI price indexes* $P_{CCDI}{}^{y*}$ are defined as follows:

(78) $P_{CCDI}{}^{y*} \equiv p_{CCDI}{}^{y*}/p_{CCDI}{}^{1*}$; $\quad y = 1,\ldots,Y.$

The 10 maximum overlap annual indexes $P_{LFB}{}^{y*}$, $P_{PFB}{}^{y*}$, $P_{FFB}{}^{y*}$, $P_{TFB}{}^{y*}$, $P_{LCH}{}^{y*}$, $P_{PCH}{}^{y*}$, $P_{FCH}{}^{y*}$, $P_{TCH}{}^{y*}$, $P_{GEKS}{}^{y*}$ and $P_{CCDI}{}^{y*}$ for our empirical example are listed in Table 12 below.

Our final annual Mudgett Stone annual index that uses year over year maximum overlap prices is the Predicted Share Similarity linked index $P_S{}^{y*}$.

Using our zero prices $p_{y,m,n}$ for products n that are not available in month m of year y, the year y, month m, product n actual expenditure share is $s_{y,m,n} \equiv p_{y,m,n}q_{y,m,n}/p^{y,m}\cdot q^{y,m}$. The *prediction* for this share using the price of product n of month m in year z, $p_{z,m,n}$, and the actual quantity of product n for month m in year y is the *predicted share* $s_{z,y,m,n} \equiv p_{z,m,n}q_{y,m,n}/p^{z,m}\cdot q^{y,m}$ for n = 1,...,N, m = 1,...,M, z = 1,...,Y and y = 1,...,Y. Using these prices and shares, the new annual measure of *Predicted Share Price Dissimilarity* between the prices of years z and y, $\Delta_{PSA}{}^*(p^z,p^y,q^z,q^y)$, is defined as follows:

(79) $\Delta_{PSA}{}^*(p^z,p^y,q^z,q^y) \equiv \Sigma_{m=1}{}^{M}\, \Sigma_{n=1}{}^{N}\, [s_{y.m,n} - s_{z,y,m,n}]^2 + \Sigma_{m=1}{}^{M}\, \Sigma_{n=1}{}^{N}\, [s_{z.m,n} - s_{y,z,m,n}]^2.$

Note that this measure of relative price dissimilarity does not use any imputed prices.[41]

The table of the new bilateral measures of Annual Predicted Share Price Dissimilarity for our empirical example is Table 11 listed below.

Table 11: Imputation Free Annual Index Predicted Share Measures of Price Dissimilarity

	y = 1	y = 2	y = 3	y = 4	y = 5	y = 6
z = 1	0.00000	0.00284	0.00272	0.00198	0.00272	0.00245
z = 2	0.00284	0.00000	0.00125	0.00275	0.00122	0.00305
z = 3	0.00272	0.00125	0.00000	0.00181	0.00086	0.00148
z = 4	0.00198	0.00275	0.00181	0.00000	0.00213	0.00056
z = 5	0.00272	0.00122	0.00086	0.00213	0.00000	0.00154
z = 6	0.00245	0.00305	0.00148	0.00056	0.00154	0.00000

A comparison of the entries in Tables 9 and 11 shows that the entries in Table 11 are always equal to or greater than the corresponding entries in Table 9. Many entries in Table 11 are substantially greater. This is due to the fact that the new measure of relative price dissimilarity that uses 0 values for missing prices instead of carry forward prices substantially penalizes a lack of matching.

The *real time* set of bilateral links which minimize the new predicted share measures of relative price dissimilarity for the annual data are as follows: link 2 to 1; 3 to 2; 4 to 3; 5 to 3 and 6 to 4. This is the same set of bilateral links that we used to construct the similarity linked annual indexes $P_S{}^y$ that used carry forward prices. Thus we define $P_S{}^{1*} \equiv 1$, $P_S{}^{2*} \equiv P_F{}^*(2/1)$, $P_S{}^{3*} \equiv P_S{}^{2*} \times P_F{}^*(3/2)$, $P_S{}^{4*} \equiv P_S{}^{3*} \times P_F{}^*(4/3)$, $P_S{}^{5*} \equiv P_S{}^{3*} \times P_F{}^*(5/3)$ and $P_S{}^{6*} \equiv P_S{}^{4*} \times P_F{}^*(6/4)$ where the maximum overlap bilateral annual Fisher indexes $P_F{}^*(y/z)$ are defined by definitions (63).

The 11 annual indexes that use maximum overlap bilateral indexes to link the months, $P_{LFB}{}^{y*}$, $P_{PFB}{}^{y*}$, $P_{FFB}{}^{y*}$, $P_{TFB}{}^{y*}$, $P_{LCH}{}^{y*}$, $P_{PCH}{}^{y*}$, $P_{FCH}{}^{y*}$, $P_{TCH}{}^{y*}$, $P_{GEKS}{}^{y*}$, $P_{CCDI}{}^{y*}$ and $P_S{}^{y*}$ are listed in Table 12 below.

Table 12: Alternative Annual Mudgett Stone Indexes Using Maximum Overlap Bilateral Indexes

Year y	$P_{LFB}{}^{y*}$	$P_{PFB}{}^{y*}$	$P_{LCH}{}^{y*}$	$P_{PCH}{}^{y*}$	$P_{FFB}{}^{y*}$	$P_{FCH}{}^{y*}$	$P_{TFB}{}^{y*}$	$P_{TCH}{}^{y*}$	$P_{GEKS}{}^{y*}$	$P_{CCDI}{}^{y*}$	$P_S{}^{y*}$
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.1373	1.0611	1.1373	1.0611	1.0986	1.0986	1.0921	1.0921	1.0964	1.0930	1.0986
3	1.1273	1.0750	1.1545	1.0468	1.1009	1.0994	1.0986	1.0929	1.0987	1.0958	1.0994
4	1.1919	1.1407	1.2419	1.0985	1.1660	1.1680	1.1633	1.1609	1.1683	1.1650	1.1680
5	1.2253	1.1565	1.2848	1.0962	1.1904	1.1868	1.1876	1.1792	1.1916	1.1881	1.1947
6	1.2344	1.1752	1.3194	1.0973	1.2044	1.2032	1.2031	1.1961	1.2056	1.2028	1.2053
Mean	1.1527	1.1014	1.1897	1.0666	1.1267	1.1260	1.1241	1.1202	1.1268	1.1242	1.1277

[41] However, one could argue that setting the price of a product that is not purchased in a period equal to 0 is also an imputation. Note that definition (79) is exactly the same as definition (60) in the previous section. But the previous definition used carry forward (and backward) prices for missing prices whereas in this section, missing prices are set equal to 0. The actual shares of product n in month m of year y, $s_{y,m.n}$, are the same in definitions (60) and (79) but the predicted share shares $s_{y,z,m,n} = p_{z,m,n}q_{y,m,n}/p^{z,m}\cdot q^{y,m}$ are now, in general, different due to the replacement of carry forward prices by zero prices.

As was the case for the Laspeyres and Paasche indexes that used carry forward prices, the new maximum overlap annual fixed base and chained Laspeyres indexes, P_{LFB}^{y*} and P_{LCH}^{y*}, are well above the superlative indexes and the new maximum overlap annual fixed base and chained Paasche indexes, P_{PFB}^{y*} and P_{PCH}^{y*}, are well below the superlative indexes. Our five best indexes are the fixed base Fisher and Törnqvist Theil indexes and the multilateral GEKS, CCDI and Predicted Share Price Similarity linked indexes. These five indexes ended up at 1.2044, 1.2031, 1.2056, 1.2028 and 1.2053. The average of these five final values is 1.2048. The average of the five final values for the same indexes listed in Table 10 is 1.2032. Thus the differences between our best maximum overlap indexes listed in Table 12 and the counterpart indexes listed in Table 10 that used carry forward prices are not large for our empirical example. The downward bias resulting from the use of carry forward prices over the sample period is only about 0.16 percentage points. However, this bias is not negligible and can be avoided by using bilateral maximum overlap indexes.

We conclude this section on annual indexes by looking at some approximations to the "true" Mudgett Stone indexes P_{LFB}^{y*}, P_{PFB}^{y*}, P_{FFB}^{y*}, P_{GEKS}^{y*}, P_{S}^{y*} that are listed in Table 12. In section 3, year over year monthly indexes were computed using bilateral maximum overlap indexes as building blocks. In particular, the fixed base Laspeyres, Paasche and Fisher indexes, $P_{LFB}^{y,m*}$, $P_{PFB}^{y,m*}$ and $P_{FFB}^{y,m*}$, were computed along with the maximum overlap GEKS index and the Predicted Share Similarity linked indexes, $P_{GEKS}^{y,m*}$ and $P_{S}^{y,m*}$. Some statistical agencies form annual indexes by taking equally weighted averages of their month to month indexes. In the previous section, we saw that the true Mudgett Stone annual Laspeyres index (using carry forward prices for missing prices) could be computed as a share weighted average of the monthly year over year indexes. It is of interest to see how taking simple equally weighted averages of the monthly indexes $P_{LFB}^{y,m*}$, $P_{PFB}^{y,m*}$, $P_{FFB}^{y,m*}$, $P_{GEKS}^{y,m*}$, $P_{S}^{y,m*}$ can approximate the "true" Mudgett Stone indexes P_{LFB}^{y*}, P_{PFB}^{y*}, P_{FFB}^{y*}, P_{GEKS}^{y*}, P_{S}^{y*}. Thus define the following *approximate annual indexes* P_{LFBA}^{y*}, P_{PFBA}^{y*}, P_{FFBA}^{y*}, P_{GEKSA}^{y*} and P_{SA}^{y*} for y = 1,...,Y as follows:

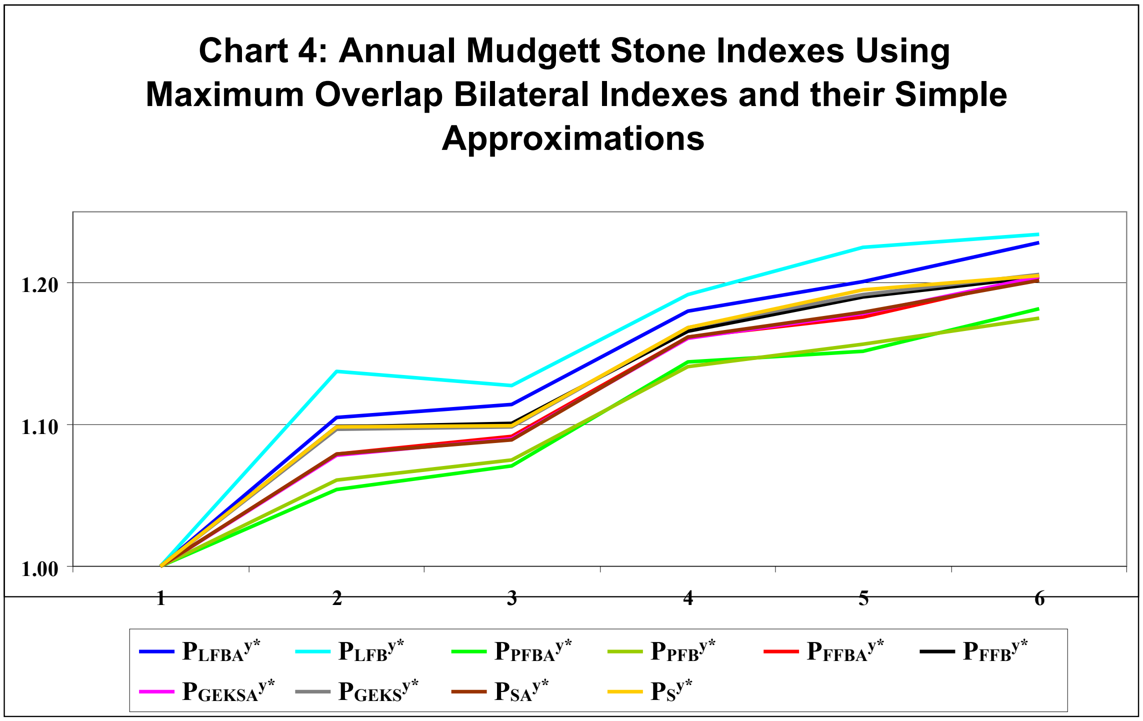
(80) $P_{LFBA}^{y*} \equiv (1/M)\Sigma_{m=1}^{M} P_{LFB}^{y,m*}$;
(81) $P_{PFBA}^{y*} \equiv (1/M)\Sigma_{m=1}^{M} P_{PFB}^{y,m*}$;
(82) $P_{FFBA}^{y*} \equiv (1/M)\Sigma_{m=1}^{M} P_{FFB}^{y,m*}$;
(83) $P_{GEKSA}^{y*} \equiv (1/M)\Sigma_{m=1}^{M} P_{GEKS}^{y,m*}$;
(84) $P_{SA}^{y*} \equiv (1/M)\Sigma_{m=1}^{M} P_{S}^{y,m*}$.

The five "true" annual indexes P_{LFB}^{y*}, P_{PFB}^{y*}, P_{FFB}^{y*}, P_{GEKS}^{y*}, P_{S}^{y*} and their five approximations P_{LFBA}^{y*}, P_{PFBA}^{y*}, P_{FFBA}^{y*}, P_{GEKSA}^{y*} and P_{SA}^{y*} evaluated using our Israeli data are listed in Table 13 below.

Table 13: Annual Mudgett Stone Indexes Using Maximum Overlap Bilateral Indexes and their Year over Year Simple Approximations

Year y	P_{LFBA}^{y*}	P_{LFB}^{y*}	P_{PFBA}^{y*}	P_{PFB}^{y*}	P_{FFBA}^{y*}	P_{FFB}^{y*}	P_{GEKSA}^{y*}	P_{GEKS}^{y*}	P_{SA}^{y*}	P_{S}^{y*}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.1053	1.1373	1.0538	1.0611	1.0789	1.0986	1.0785	1.0964	1.0789	1.0986
3	1.1141	1.1273	1.0706	1.0750	1.0920	1.1009	1.0902	1.0987	1.0896	1.0994
4	1.1802	1.1919	1.1438	1.1407	1.1617	1.1660	1.1612	1.1683	1.1614	1.1680
5	1.2012	1.2253	1.1520	1.1565	1.1761	1.1904	1.1785	1.1916	1.1788	1.1947
6	1.2279	1.2344	1.1817	1.1752	1.2045	1.2044	1.2040	1.2056	1.2014	1.2053
Mean	1.1381	1.1527	1.1003	1.1014	1.1189	1.1267	1.1187	1.1268	1.1184	1.1277

Chart 4 below plots the above 10 indexes.



As usual, the two fixed base Laspeyres indexes are well above the superlative indexes and the two fixed base Paasche indexes are well below the superlative indexes. What is interesting is that the approximate Laspeyres indexes $P_{LFBA}{}^{y*}$ lie well above their "true" counterparts, $P_{LFB}{}^{y*}$. Moreover, there are some substantial differences in the average values for the "true" superlative indexes and their approximations. The average for the true fixed base Fisher annual indexes $P_{FFB}{}^{y*}$ is 1.1267 which is well above the average for the approximate fixed base Fisher indexes $P_{FFBA}{}^{y*}$ which is 1.1189. The average for the true similarity linked Fisher indexes $P_{S}{}^{y*}$ is 1.1277 which is well above the average for the approximate similarity linked Fisher indexes $P_{SA}{}^{y*}$ which is 1.1184. The average for the true GEKS annual indexes $P_{GEKS}{}^{y*}$ is 1.1268 which is also above the average for the GEKS approximate indexes $P_{GEKSA}{}^{y*}$ which is 1.1187.

Our conclusions regarding the construction of annual indexes at this point are as follows:

- The use of the Laspeyres and Paasche Mudgett Stone indexes should be avoided. The fixed base and chained Laspeyres indexes tend to lie well above the clustered superlative indexes while the fixed base and chained Paasche indexes tend to lie well below the clustered superlative indexes.
- The amount of chain drift in the annual Fisher and Törnqvist Theil indexes was small for our empirical example. However, if one used the similarity linked annual Mudgett Stone indexes, there is no possibility of any chain drift.
- The Mudgett Stone fixed base Fisher and Törnqvist Theil indexes and the GEKS and Predicted Share Similarity linked indexes are all fairly close to each other in the present context where we are calculating annual indexes.
- The use of carry forward prices will tend to lead to annual indexes which are biased downward if there is general inflation and so in order to avoid this potential bias, it is better to use the indexes that use maximum overlap superlative bilateral indexes as their

basic building blocks. Thus the maximum overlap annual fixed base Fisher and fixed base Törnqvist Theil, the GEKS and the Predicted Share similarity linked indexes, $P_{FFB}{}^{y,*}$, $P_{TFB}{}^{y*}$, $P_{GEKS}{}^{y*}$ and $P_S{}^{y*}$, emerge as our "best" choices for Mudgett Stone annual indexes.

- Approximating "true" Mudgett Stone indexes by taking a simple average of the year over year monthly indexes discussed in sections 2 and 3 can lead to substantial approximation errors. For our empirical example, the approximation error using the Laspeyres formula was substantial.

In the following sections, we turn our attention to month to month price indexes.

6. Month to Month Indexes using Carry Forward Prices

Some new notation is required when constructing month to month indexes for seasonal goods and services. Denote the *quantity purchased* of product n in month t as $q_{t,n}$ where t = 1,2,...,T where T = MY, M denotes the number of months for the data set under consideration and Y denotes the number of years of seasonal product data. Thus t is now a monthly time indicator which runs from 1 to T. As usual, if no units of product n are purchased in month t, $q_{t,n} = 0$. If product n is purchased in month t, then denote the corresponding *unit value price* for this product by $p_{t,n} > 0$ for n = 1,...,N and t = 1,...,T. In this section, if product n is missing in month t, then $p_{t,n}$ is set equal to the most recent previous month price for product n; i.e., in this section, we replace missing prices by *month to month carry forward prices*. If product n is missing in month 1, then $p_{1,n}$ is set equal to the price of product n in the next month when the product is sold; i.e., in this case, we use a *month to month carry backward* price for $p_{1,n}$. In general, these carry forward and carry backward prices will be substantially different from the carry forward and backward prices which were used in sections 2 and 4 above. The frequency of imputed prices greatly increases when constructing price indexes for strongly seasonal commodities. For our empirical example, there were 451 month to month carry forward or carry backward prices where the maximum number of available products over the months in our sample was 1008 = 72 months × 14 fresh fruit products. Tables A23 and A24 in the Appendix list the price and quantity data for fresh fruit purchased by households in Israel for the 72 months in the years 2012-2017. The sample probability that a price listed in Table A23 is an imputed price is 0.447 = 451/1008. *Thus the problem of missing prices can be a very big problem in the seasonal product context.*

In this section, we will set up the algebra for computing fixed base and chained Laspeyres, Paasche and Fisher month to month indexes using carry forward/backward prices for unavailable products. The monthly price and quantity variables, $p_{t,n}$ and $q_{t,n}$ for product n in month t have been defined in the previous paragraph. Define the month t vectors of product prices and quantities, p^t and q^t as $p^t \equiv [p_{t,1},...,p_{t,N}]$ and $q^t \equiv [q_{t,1},...,q_{t,N}]$. For our empirical example, T = 72 and N = 14.

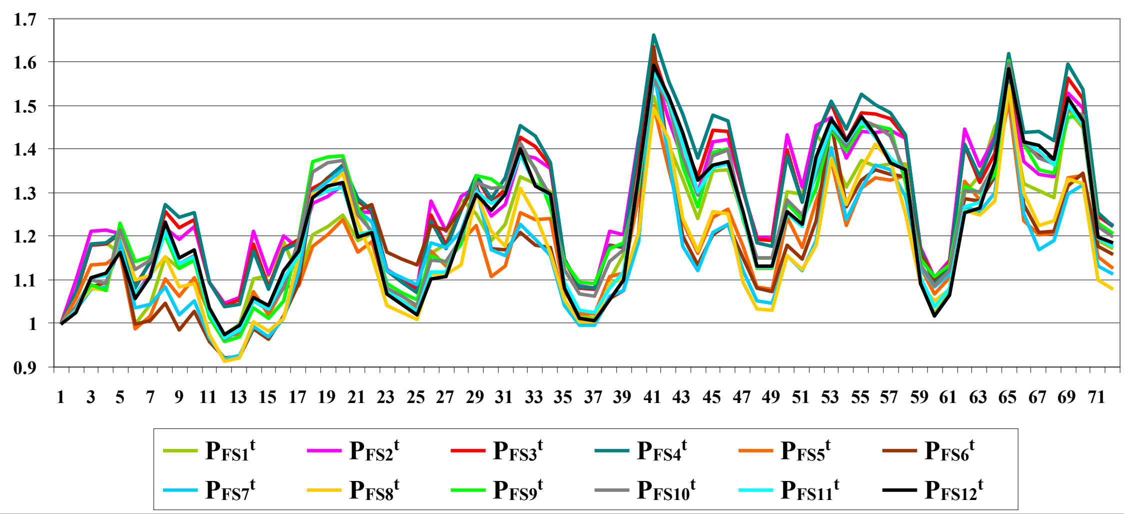
Denote the *bilateral Laspeyres, Paasche and Fisher price indexes* that compare the prices of month t relative to the prices of month r using carry forward/backward prices as $P_L(t/r)$, $P_P(t/r)$ and $P_F(t/r)$ respectively. These indexes are defined as follows:

(85) $P_L(t/r) \equiv p^t \cdot q^r / p^r \cdot q^r$; $\quad r = 1,...,T;\ t = 1,...,T;$
(86) $P_P(t/r) \equiv p^t \cdot q^t / p^r \cdot q^t$; $\quad r = 1,...,T;\ t = 1,...,T;$
(87) $P_F(t/r) \equiv [P_L(t/r)P_P(t/r)]^{1/2}$; $\quad r = 1,...,T;\ t = 1,...,T.$

The sequence of T *fixed base Laspeyres indexes* using carry forward prices, $P_{LFB}{}^t$, is $P_L(1/1)$, $P_L(2/1)$, ..., $P_L(T/1)$. The sequence of T *fixed base Paasche indexes* using carry forward prices, $P_{PFB}{}^t$, is $P_P(1/1)$, $P_P(2/1)$, ..., $P_P(T/1)$ and the sequence of T *fixed base Fisher indexes* using carry forward prices, $P_{FFB}{}^t$, is $P_F(1/1)$, $P_F(2/1)$, ..., $P_F(T/1)$. We use the data listed in Tables A23 and A24 in the Appendix to calculate these indexes for our Israeli data set. These indexes are listed in Table 15 below.

It should be noted that the month to month indexes defined by (85)-(87) are not very reliable for our empirical example. Here is a list of the number of seasonal products that are actually available in months 1-12: 7, 8, 8, 7, 9, 10, 8, 7, 7, 10, 9, 7. The maximum number of products is 14. Thus for 5 out of the first 12 months, only one half of the seasonal fruits are available. When we look at matches for the products that are available in both month 1 and month m = 1,...,12, we find that the number of product matches is 7, 7, 7, 6, 5, 5, 3, 3, 4, 7, 7, 7. Because of the low number of bilateral product matches, we cannot expect any index number to be very reliable.

Chart 5: Fisher Star Indexes Using Months 1-12 as the Base and Using Carry Forward Prices



Instead of choosing month 1 to be the fixed base, we could chose any other month as the fixed base. The resulting indexes are called "star" indexes. The 12 fixed base Fisher star indexes using months 1-12 as the base month are listed in Table A25 of the Appendix and are plotted on Chart 5 above. These indexes have been normalized to equal 1 in month 1.

A number of points emerge from a study of Chart 5:

- The seasonal fluctuations in prices are enormous;
- The choice of a base period matters;
- Any monthly index number is unlikely to be very reliable for our particular data set.

The problems associated with the reliability of month to month indexes of strongly seasonal commodities are much bigger than the problem of finding reliable year over year monthly indexes. As was seen in the previous sections, our best year over year monthly indexes were well behaved and approximated each other fairly well. This is not the case for month to month indexes.

Define the *month to month chained Laspeyres, Paasche* and *Fisher indexes using carry forward prices* for month 1 as unity:

(88) $P_{LCH}^1 \equiv 1$; $P_{PCH}^1 \equiv 1$; $P_{FCH}^1 \equiv 1$.

For months following month 1, these chained indexes for month t are calculated by cumulating the corresponding successive month to month links using definitions (85)-(88); ; i.e., we have the following definitions for P_{LCH}^t, P_{PCH}^t and P_{FCH}^t:

(89) $P_{LCH}^t \equiv P_{LCH}^{t-1}\, P_L(t/(t-1))$; $\quad t = 2,3,...,T;$
(90) $P_{PCH}^t \equiv P_{PCH}^{t-1}\, P_P(t/(t-1))$; $\quad t = 2,3,...,T;$
(91) $P_{FCH}^t \equiv P_{FCH}^{t-1}\, P_F(t/(t-1))$; $\quad t = 2,3,...,T.$

The *month to month GEKS price levels* using carry forward prices, p_{GEKS}^t, for each month t is defined as the geometric mean of the separate indexes we obtain by using each month as the base year:

(92) $p_{GEKS}^t \equiv [\prod_{r=1}^T P_F(t/r)]^{1/T}$; $\quad t = 1,2,...,T$

where $P_F(t/r)$ is defined by (87). The *month to month GEKS price indexes* P_{GEKS}^t are obtained by *normalizing* the above price levels so that the month 1 index is equal to 1. Thus we have the following definitions for the *GEKS month to month index using carry forward prices* for month t:

(93) $P_{GEKS}^t \equiv p_{GEKS}^t/p_{GEKS}^1$; $\quad t = 1,2,...,T.$

The month to month GEKS indexes using carry forward prices along with the chained month to month Laspeyres, Paasche and Fisher indexes for our Israeli data are listed below in Table 15.

The final month to month index that we define in this section is the *Predicted Share Similarity linked index*, P_S^t. The *month t, product n actual expenditure share* is $s_{t,n}$ defined as follows:

(94) $s_{t,n} \equiv p_{t,n}q_{t,n}/p^t\bullet q^t$; $\quad t = 1,...,T;\ n = 1,...,N.$

The prediction for this share $s_{t,n}$ using the price of product n in month r, $p_{r,n}$, and the actual quantity of product n in month t is the *predicted share* $s_{r,t,n} \equiv p_{r,n}q_{t,n}/p^r\bullet q^t$ for $n = 1,...,N$, $r = 1,...,T$, $t = 1,...,T$. The new measure of *month to month Predicted Share Price Dissimilarity* between the prices of months r and t, $\Delta_{PS}(p^r,p^t,q^r,q^t)$, is defined as follows:

(95) $\Delta_{PS}(p^r,p^t,q^r,q^t) \equiv \Sigma_{n=1}^N [s_{t,n} - s_{r,t,n}]^2 + \Sigma_{n=1}^N [s_{r,n} - s_{t,r,n}]^2$; $\quad r = 1,...,T;\ t = 1,...,T.$

The entire set of predicted share dissimilarity measures for our empirical example is a 72 by 72 element (symmetric) matrix. Table 14 below lists the first 12 rows and columns of the matrix of the bilateral measures of Annual Predicted Share Price Dissimilarity for our empirical example.

Table 14: Month to Month Predicted Share Measures of Price Dissimilarity Using Carry Forward Prices

r,t	1	2	3	4	5	6	7	8	9	10	11	12
1	0.0000	0.0570	0.0287	0.0074	0.0700	0.5867	0.5816	0.2707	0.0971	0.0303	0.0194	0.0103
2	0.0570	0.0000	0.0161	0.1042	0.0043	0.1083	0.0916	0.0440	0.0316	0.0306	0.0318	0.0621
3	0.0287	0.0161	0.0000	0.0370	0.0078	0.1519	0.1309	0.0554	0.0179	0.0130	0.0086	0.0195

4	**0.0074**	**0.1042**	**0.0370**	**0.0000**	**0.0601**	**0.4277**	**0.4049**	**0.1784**	**0.0576**	**0.0201**	**0.0225**	**0.0149**
5	**0.0700**	**0.0043**	**0.0078**	**0.0601**	**0.0000**	**0.1239**	**0.1134**	**0.0423**	**0.0280**	**0.0146**	**0.0112**	**0.0237**
6	**0.5867**	**0.1083**	**0.1519**	**0.4277**	**0.1239**	**0.0000**	**0.0038**	**0.0408**	**0.0967**	**0.1145**	**0.1586**	**0.2551**
7	**0.5816**	**0.0916**	**0.1309**	**0.4049**	**0.1134**	**0.0038**	**0.0000**	**0.0221**	**0.0678**	**0.0906**	**0.1401**	**0.2392**
8	**0.2707**	**0.0440**	**0.0554**	**0.1784**	**0.0423**	**0.0408**	**0.0221**	**0.0000**	**0.0136**	**0.0244**	**0.0495**	**0.0972**
9	**0.0971**	**0.0316**	**0.0179**	**0.0576**	**0.0280**	**0.0967**	**0.0678**	**0.0136**	**0.0000**	**0.0069**	**0.0219**	**0.0472**
10	**0.0303**	**0.0306**	**0.0130**	**0.0201**	**0.0146**	**0.1145**	**0.0906**	**0.0244**	**0.0069**	**0.0000**	**0.0053**	**0.0112**
11	**0.0194**	**0.0318**	**0.0086**	**0.0225**	**0.0112**	**0.1586**	**0.1401**	**0.0495**	**0.0219**	**0.0053**	**0.0000**	**0.0017**
12	**0.0103**	**0.0621**	**0.0195**	**0.0149**	**0.0237**	**0.2551**	**0.2392**	**0.0972**	**0.0472**	**0.0112**	**0.0017**	**0.0000**

The set of real time links which minimize the above dissimilarity measures for the first 12 observations are as follows:

```
4   5
|   |
1 – 2 – 3
    |
    6 – 7 – 8 – 9 – 10 – 11 – 12
```

It can be seen that there are substantial differences in the measures of relative price dissimilarity across pairs of observations. If any measure not on the main diagonal of the matrix of dissimilarity measures is equal to zero, then prices are proportional for the corresponding pair of months. It can be seen that for months 11 and 12, the dissimilarity measure is 0.0017 so that prices are "almost" proportional to each other for that pair of months.

The real time *month to month Predicted Share indexes* for months 1 to 12 are defined as follows. $P_S^1 \equiv 1$; $P_S^2 \equiv P_F(2/1)$ where the bilateral Fisher indexes $P_F(t/r)$ are defined by (87). $P_S^3 \equiv P_F(3/2)P_S^2$; $P_S^4 \equiv P_F(4/1)P_S^1$; $P_S^5 \equiv P_F(5/2)P_S^2$; $P_S^6 \equiv P_F(6/2)P_S^2$; $P_S^7 \equiv P_F(7/6)P_S^6$; $P_S^8 \equiv P_F(8/7)P_S^7$; $P_S^9 \equiv P_F(9/8)P_S^8$; $P_S^{10} \equiv P_F(10/9)P_S^9$; $P_S^{11} \equiv P_F(11/10)P_S^{10}$; $P_S^{12} \equiv P_F(12/11)P_S^{11}$.[42]

The Predicted Share indexes (using carry forward prices) P_S^t along with the other 7 indexes defined in this section are listed below in Table 15.

Table 15: Alternative Month to Month Price Indexes Using Carry Forward Prices

t	P_{LFB}^t	P_{LCH}^t	P_{PFB}^t	P_{PCH}^t	P_{FCH}^t	P_{FFB}^t	P_{GEKS}^t	P_S^t
1	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
2	**1.07104**	**1.07104**	**1.12853**	**1.12853**	**1.09941**	**1.09941**	**1.04977**	**1.09941**
3	**1.12812**	**1.18503**	**1.23293**	**1.24226**	**1.21331**	**1.17936**	**1.12214**	**1.21331**
4	**1.14886**	**1.19038**	**1.22398**	**1.24277**	**1.21629**	**1.18583**	**1.12065**	**1.18583**
5	**1.18497**	**1.19577**	**1.13834**	**1.28803**	**1.24105**	**1.16142**	**1.20425**	**1.20614**
6	**1.13858**	**1.05378**	**0.87546**	**1.05587**	**1.05482**	**0.99839**	**1.03327**	**1.06618**
7	**1.21631**	**1.08372**	**0.89541**	**1.04168**	**1.06249**	**1.04360**	**1.09091**	**1.07394**
8	**1.42856**	**1.13888**	**0.93258**	**1.07104**	**1.10444**	**1.15423**	**1.20334**	**1.11633**
9	**1.30179**	**1.08934**	**0.97666**	**0.98544**	**1.03609**	**1.12756**	**1.10585**	**1.04725**
10	**1.23076**	**1.11710**	**1.08481**	**0.99017**	**1.05172**	**1.15549**	**1.13253**	**1.06305**
11	**1.03294**	**1.03012**	**1.01778**	**0.84758**	**0.93440**	**1.02533**	**1.01883**	**0.94447**
12	**0.97081**	**0.97490**	**0.98105**	**0.79429**	**0.87997**	**0.97592**	**0.96872**	**0.88945**

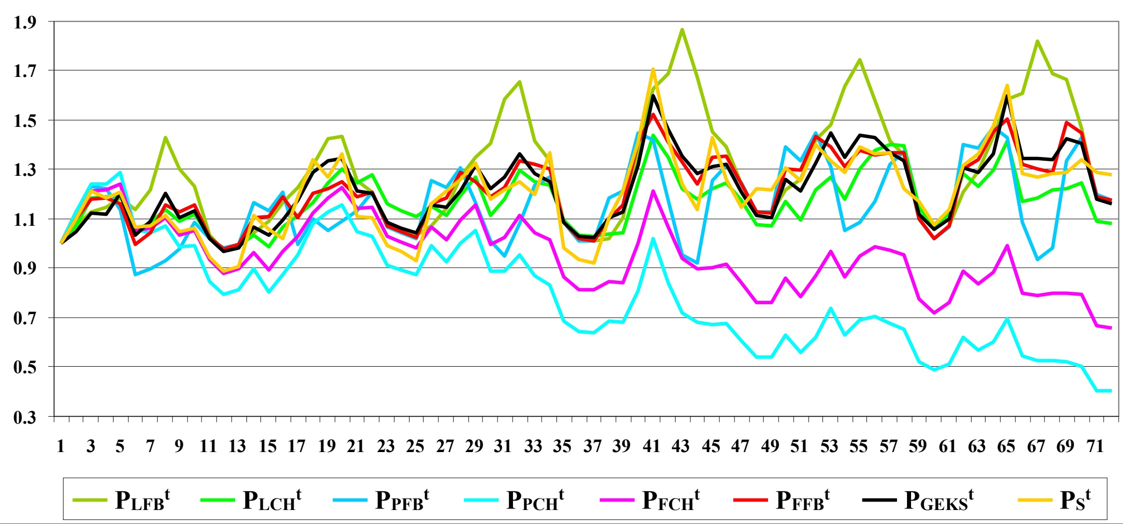
[42] The optimal real time bilateral Fisher index links for the next 12 months are as follows: 13/12, 14/5, 15/5, 16/12, 17/15, 18/7, 19/8, 20/17, 21/11, 22/13, 23/11 and 24/23. The optimal links are usually to an adjacent month or to the same month in a previous year. Thus the bilateral links for the relative price similarity linked indexes are a mixture of chain links and year over year links (or almost year over year links).

13	**0.99746**	**0.99246**	**0.99881**	**0.81404**	**0.89884**	**0.99813**	**0.98116**	**0.90852**
14	**1.04362**	**1.03468**	**1.16672**	**0.89817**	**0.96401**	**1.10345**	**1.06726**	**1.11577**
15	**1.08902**	**0.98580**	**1.13349**	**0.80588**	**0.89131**	**1.11103**	**1.03202**	**1.05922**
16	**1.16801**	**1.06661**	**1.20920**	**0.87626**	**0.96676**	**1.18842**	**1.09357**	**1.02178**
17	**1.22562**	**1.11251**	**0.99841**	**0.95309**	**1.02972**	**1.10619**	**1.17701**	**1.18894**
18	**1.31761**	**1.16525**	**1.09946**	**1.08230**	**1.12301**	**1.20361**	**1.28605**	**1.33771**
19	**1.42276**	**1.24559**	**1.05160**	**1.12761**	**1.18513**	**1.22318**	**1.33392**	**1.26846**
20	**1.43404**	**1.30259**	**1.09058**	**1.15575**	**1.22698**	**1.25057**	**1.34386**	**1.36295**
21	**1.25621**	**1.24446**	**1.13062**	**1.04605**	**1.14095**	**1.19176**	**1.21098**	**1.10877**
22	**1.20769**	**1.27816**	**1.20617**	**1.03098**	**1.14794**	**1.20693**	**1.20203**	**1.10482**
23	**1.07109**	**1.15892**	**1.06924**	**0.91408**	**1.02925**	**1.07017**	**1.08536**	**0.98947**
24	**1.05248**	**1.13311**	**1.04479**	**0.89146**	**1.00505**	**1.04863**	**1.06297**	**0.96621**
25	**1.03276**	**1.10994**	**1.01894**	**0.87344**	**0.98461**	**1.02583**	**1.04160**	**0.93097**
26	**1.07388**	**1.15260**	**1.25452**	**0.99093**	**1.06871**	**1.16069**	**1.15761**	**1.16265**
27	**1.14208**	**1.11185**	**1.22626**	**0.92749**	**1.01549**	**1.18342**	**1.14743**	**1.21314**
28	**1.26758**	**1.19400**	**1.30863**	**1.00128**	**1.09340**	**1.28794**	**1.21375**	**1.25992**
29	**1.34863**	**1.26883**	**1.16368**	**1.05079**	**1.15467**	**1.25275**	**1.31550**	**1.32748**
30	**1.40760**	**1.11401**	**1.00832**	**0.88728**	**0.99420**	**1.19135**	**1.22045**	**1.18155**
31	**1.58269**	**1.18055**	**0.95120**	**0.89015**	**1.02512**	**1.22697**	**1.26719**	**1.21830**
32	**1.65416**	**1.29680**	**1.08119**	**0.95584**	**1.11334**	**1.33733**	**1.36536**	**1.24880**
33	**1.41549**	**1.24967**	**1.23415**	**0.86813**	**1.04157**	**1.32172**	**1.28388**	**1.19893**
34	**1.33751**	**1.23520**	**1.26955**	**0.83316**	**1.01445**	**1.30308**	**1.25193**	**1.36985**
35	**1.08703**	**1.09589**	**1.08493**	**0.68413**	**0.86587**	**1.08598**	**1.08582**	**0.98455**
36	**1.02305**	**1.03395**	**1.01285**	**0.64242**	**0.81500**	**1.01793**	**1.02783**	**0.93662**
37	**1.01159**	**1.02930**	**1.00992**	**0.63836**	**0.81060**	**1.01076**	**1.02600**	**0.92221**
38	**1.02156**	**1.03700**	**1.18691**	**0.68755**	**0.84438**	**1.10114**	**1.10446**	**1.10545**
39	**1.10562**	**1.04371**	**1.21361**	**0.68105**	**0.84310**	**1.15836**	**1.12719**	**1.21204**
40	**1.37534**	**1.24541**	**1.44936**	**0.80753**	**1.00285**	**1.41186**	**1.31496**	**1.41186**
41	**1.62925**	**1.43926**	**1.42127**	**1.02153**	**1.21254**	**1.52171**	**1.59969**	**1.70708**
42	**1.68676**	**1.35068**	**1.17617**	**0.84241**	**1.06669**	**1.40851**	**1.46018**	**1.42062**
43	**1.86492**	**1.22188**	**0.95194**	**0.71973**	**0.93778**	**1.33240**	**1.35535**	**1.24298**
44	**1.67566**	**1.18164**	**0.91940**	**0.67888**	**0.89565**	**1.24121**	**1.28485**	**1.13731**
45	**1.45074**	**1.22157**	**1.25527**	**0.67003**	**0.90470**	**1.34947**	**1.31324**	**1.43102**
46	**1.39276**	**1.24584**	**1.31660**	**0.67430**	**0.91656**	**1.35414**	**1.32060**	**1.25905**
47	**1.24213**	**1.17629**	**1.24472**	**0.60421**	**0.84304**	**1.24342**	**1.20656**	**1.14893**
48	**1.12808**	**1.07490**	**1.12696**	**0.54158**	**0.76298**	**1.12752**	**1.11280**	**1.22226**
49	**1.12212**	**1.07078**	**1.12896**	**0.53941**	**0.75999**	**1.12554**	**1.10687**	**1.21747**
50	**1.21916**	**1.16985**	**1.39205**	**0.62882**	**0.85768**	**1.30274**	**1.26213**	**1.30932**
51	**1.25881**	**1.09697**	**1.33488**	**0.55975**	**0.78360**	**1.29628**	**1.21290**	**1.25639**
52	**1.41954**	**1.21707**	**1.44776**	**0.62199**	**0.87006**	**1.43358**	**1.32513**	**1.39974**
53	**1.48016**	**1.26847**	**1.31269**	**0.73565**	**0.96600**	**1.39391**	**1.44974**	**1.33393**
54	**1.63803**	**1.18186**	**1.05207**	**0.62930**	**0.86241**	**1.31276**	**1.34921**	**1.28696**
55	**1.74314**	**1.30177**	**1.08521**	**0.69280**	**0.94967**	**1.37538**	**1.43877**	**1.39240**
56	**1.58174**	**1.37926**	**1.17009**	**0.70348**	**0.98503**	**1.36043**	**1.43084**	**1.36462**
57	**1.41498**	**1.40348**	**1.32148**	**0.67760**	**0.97519**	**1.36743**	**1.37035**	**1.36621**
58	**1.35851**	**1.39696**	**1.37335**	**0.65464**	**0.95630**	**1.36591**	**1.33524**	**1.22293**
59	**1.09904**	**1.14658**	**1.09780**	**0.52293**	**0.77432**	**1.09842**	**1.11657**	**1.17281**
60	**1.02215**	**1.06443**	**1.02087**	**0.48590**	**0.71917**	**1.02151**	**1.05685**	**1.07723**
61	**1.07410**	**1.12216**	**1.06543**	**0.51332**	**0.75897**	**1.06976**	**1.09842**	**1.13684**
62	**1.20643**	**1.28194**	**1.40317**	**0.61862**	**0.89052**	**1.30108**	**1.30876**	**1.31666**
63	**1.29331**	**1.23062**	**1.38806**	**0.56870**	**0.83657**	**1.33984**	**1.29044**	**1.36207**
64	**1.43622**	**1.29686**	**1.47064**	**0.59945**	**0.88171**	**1.45333**	**1.36261**	**1.47722**
65	**1.58284**	**1.41595**	**1.42738**	**0.69306**	**0.99062**	**1.50310**	**1.59785**	**1.64269**
66	**1.60835**	**1.17240**	**1.08403**	**0.54653**	**0.80047**	**1.32042**	**1.34683**	**1.28567**
67	**1.82150**	**1.18662**	**0.93451**	**0.52576**	**0.78986**	**1.30469**	**1.34389**	**1.26863**
68	**1.68998**	**1.21567**	**0.98336**	**0.52566**	**0.79939**	**1.28913**	**1.34143**	**1.28556**
69	**1.66533**	**1.22295**	**1.33677**	**0.52324**	**0.79993**	**1.49203**	**1.42418**	**1.28643**
70	**1.46701**	**1.24639**	**1.42983**	**0.50438**	**0.79288**	**1.44830**	**1.40462**	**1.34180**
71	**1.18124**	**1.09020**	**1.19942**	**0.40539**	**0.66479**	**1.19030**	**1.17781**	**1.28946**

72	**1.17122**	**1.08049**	**1.17533**	**0.40306**	**0.65993**	**1.17327**	**1.15954**	**1.27798**
Mean	**1.30070**	**1.17250**	**1.15570**	**0.79632**	**0.95632**	**1.21950**	**1.22000**	**1.20780**

As usual, the Laspeyres and Paasche fixed base and chained indexes end up at very different destinations. The 3 chained indexes are all subject to a large amount of downward chain drift. This is due to the fact that the strongly seasonal commodities come into season at relatively high prices and then trend down to relatively low prices at the end of their seasonal availability. They behave in the same manner as fashion goods, which are also subject to tremendous downward chain drift.[43] Our 3 best indexes (P_{FFB}^t, P_{GEKS}^t and P_S^t) have roughly the same mean but the similarity linked index P_S^t ends up well above P_{FFB}^t and P_{GEKS}^t for t = 72. As indicated above, no index is likely to emerge as a clear winner for our particular data set due to the lack of product matching. The similarity linked indexes did not work well in this context where carry forward/backward prices are used. The use of carry forward prices causes our measures of relative price dissimilarity to be too low: there is no explicit penalty for a lack of product matching. The above 6 series are plotted in Chart 6 below.

Chart 6: Alternative Month to Month Indexes Using Carry Forward Prices



It can be seen that our 3 best indexes, P_{FFB}^t, P_{GEKS}^t and P_S^t, are much closer to each other than 4 of the other 5 indexes which suffer from substitution bias or chain drift bias.[44]

As our example indicated, the use of carry forward prices in the context of an elementary index category that includes many strongly seasonal commodities can lead to a large number of imputed

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[43] Note that the year over year monthly indexes did not suffer from this tremendous downward chain drift. Thus year over year indexes work well for both strongly seasonal goods and services as well as for fashion goods.

[44] The chained Laspeyres index ends up reasonably close to the 3 superlative indexes. It appears that the upward substitution bias (which a Laspeyres index is subject to) approximately offsets the downward chain drift bias that the chained indexes are subject to in the present context when beginning of season prices are generally higher than the corresponding end of season prices.

prices, which in turn can lead to indexes which are very different from their matched product counterpart indexes. In the following section, we will compute the maximum overlap counterpart indexes to the eight indexes listed above. This will cure any carry forward/backward bias that probably is present in the above 8 indexes.

7. Month to Month Indexes using Maximum Overlap Bilateral Indexes as Building Blocks

The month to month maximum overlap indexes that are defined in this section are analogues to the eight indexes that were defined in the previous section. The difference is that the building block bilateral indexes between periods r and t use only the prices and quantities that are actually available in periods r and t. As in the previous section, the price and quantity of product n purchased in month t is p_{tn} and q_{tn} respectively. If there are no purchases of product n in period t, set $p_{tn} = q_{tn} = 0$. Thus any missing prices are set equal to zero in this section. As usual, the set of available products in period t is denoted by S(t) for t = 1,...,T.

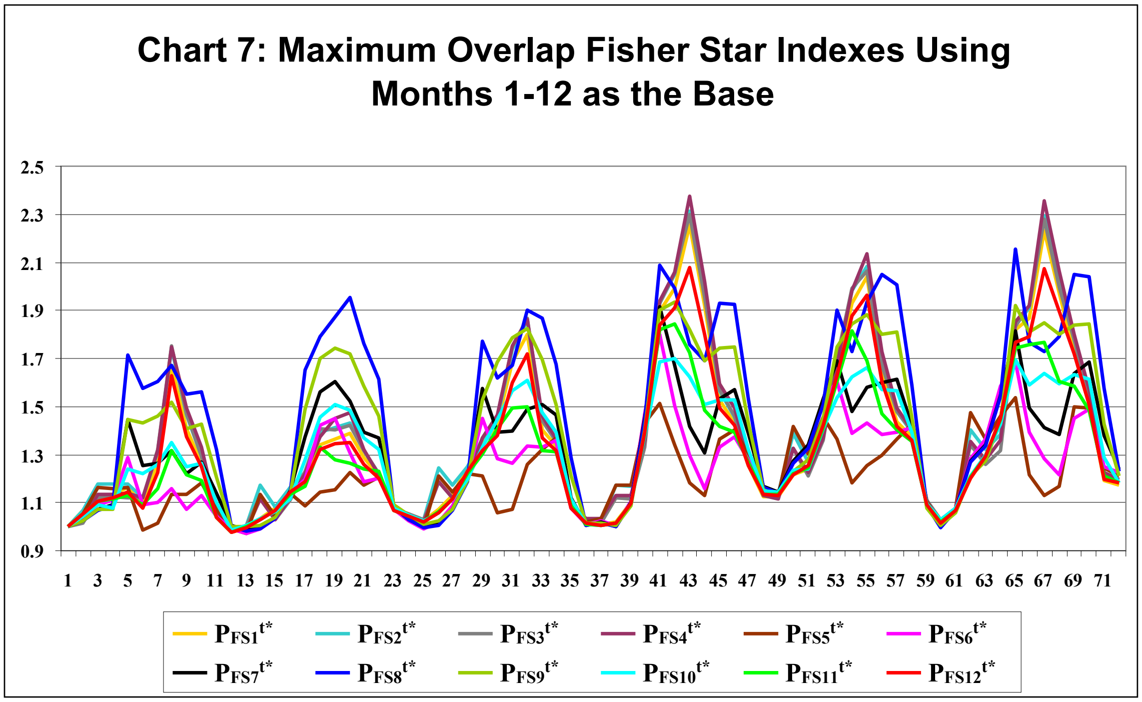
Denote the *maximum overlap bilateral Laspeyres, Paasche and Fisher price indexes* that compare the prices of month t relative to the prices of month r as $P_L^*(t/r)$, $P_P^*(t/r)$ and $P_F^*(t/r)$ respectively. These indexes are defined as follows:

$$(96)\ P_L^*(t/r) \equiv \Sigma_{n\in S(t)\cap S(r)}\ p_{t,n}q_{r,n}/\Sigma_{n\in S(t)\cap S(r)}\ p_{r,n}q_{r,n}\ ; \qquad r = 1,...,T;\ t = 1,...,T;$$
$$(97)\ P_P^*(t/r) \equiv \Sigma_{n\in S(t)\cap S(r)}\ p_{t,n}q_{t,n}/\Sigma_{n\in S(t)\cap S(r)}\ p_{r,n}q_{t,n}\ ; \qquad r = 1,...,T;\ t = 1,...,T;$$
$$(98)\ P_F^*(t/r) \equiv [P_L^*(t/r)P_P^*(t/r)]^{1/2}\ ; \qquad r = 1,...,T;\ t = 1,...,T.$$

The sequence of T *maximum overlap fixed base Laspeyres indexes*, P_{LFB}^{t*}, is $P_L^*(1/1)$, $P_L^*(2/1)$, ..., $P_L^*(T/1)$. The sequence of T *maximum overlap fixed base Paasche indexes*, P_{PFB}^{t*}, is $P_P^*(1/1)$, $P_P^*(2/1)$, ..., $P_P^*(T/1)$ and the sequence of T *maximum overlap fixed base Fisher indexes*, P_{FFB}^{t*}, is $P_F^*(1/1)$, $P_F^*(2/1)$, ..., $P_F^*(T/1)$. We use the data listed in Tables A23 and A24 in the Appendix to calculate these indexes for our Israeli data set. These indexes are listed in Table 17 below.

As in the previous section, instead of choosing month 1 to be the fixed base, we could chose any other month as the fixed base. The 12 maximum overlap fixed base Fisher star indexes using months 1-12 as the base month are listed in Table A26 of the Appendix and are plotted on Chart 7 below. These indexes have been normalized to equal 1 in month 1.

A comparison of Charts 5 and 7 shows that the use of maximum overlap fixed base Fisher indexes has led to alternative fixed base indexes which are very close to each other for the months of December, January and February but have *much larger seasonal fluctuations* than their fixed base Fisher index carry forward counterparts for other months of the year. For these alternative fixed base Fisher indexes, *the use of maximum overlap bilateral Fisher indexes has led to index values in month 72 which are on average 2.68 percentage points above their carry forward fixed base Fisher index counterparts*. Thus we have a rough estimate of the cumulative amount of downward bias that the use of carry forward prices induced for our empirical example over the six year sample period.



Define the *maximum overlap month to month chained Laspeyres, Paasche* and *Fisher indexes* for month 1 as unity:

(99) $P_{LCH}^{1*} \equiv 1$; $P_{PCH}^{1*} \equiv 1$; $P_{FCH}^{1*} \equiv 1$;

For months following month 1, these chained indexes for month t are calculated by cumulating the corresponding successive month to month links using definitions (96)-(98); ; i.e., we have the following definitions for P_{LCH}^{t*}, P_{PCH}^{t*} and P_{FCH}^{t*}:

(100) $P_{LCH}^{t*} \equiv P_{LCH}^{t-1*}\, P_L^{*}(t/(t-1))$; t = 2,3,...,T;
(101) $P_{PCH}^{t*} \equiv P_{PCH}^{t-1*}\, P_P^{*}(t/(t-1))$; t = 2,3,...,T;
(102) $P_{FCH}^{t*} \equiv P_{FCH}^{t-1*}\, P_F^{*}(t/(t-1))$; t = 2,3,...,T.

The *maximum overlap month to month GEKS price level*, p_{GEKS}^{t*}, for each month t is defined as the geometric mean of the separate maximum overlap indexes we obtain by using each month as the base year:

(103) $p_{GEKS}^{t*} \equiv [\prod_{r=1}^{T} P_F^{*}(t/r)]^{1/T}$; t = 1,2,...,T

where $P_F^{*}(t/r)$ is defined by (98). The *maximum overlap month to month GEKS price indexes* P_{GEKS}^{t*} are obtained by *normalizing* the above price levels so that the month 1 index is equal to 1. Thus we have the following definition for the month t *year over year maximum overlap GEKS index*, P_{GEKS}^{t*}:

(104) $P_{GEKS}^{t*} \equiv p_{GEKS}^{t*}/p_{GEKS}^{1*}$; t = 1,2,...,T.

The various month to month Laspeyres, Paasche, Fisher fixed base and chained indexes as well as the GEKS index defined above in this section using maximum overlap bilateral indexes as building blocks using our Israeli data are listed below in Table 17.

The final month to month index that we define in this section is the Predicted Share Similarity linked index, P_S^{t*}. Definitions (105) and (106) below are the same as definitions (94) and (95) in the previous section but in this section, the price of an unavailable product is set equal to 0. For convenience, we repeat these definitions. The *month t, product n actual expenditure share* is $s_{t,n}$ defined as follows:

$$(105)\ s_{t,n} \equiv p_{t,n}q_{t,n}/p^t\cdot q^t\ ; \qquad t = 1,...,T;\ n = 1,...,N.$$

The prediction for this share $s_{t,n}$ using the price of product n in month r, $p_{r,n}$, and the actual quantity of product n in month t is the *predicted share* $s_{r,t,n} \equiv p_{r,n}q_{t,n}/p^r\cdot q^t$ for n = 1,...,N, r = 1,...,T, t = 1,...,T. The new measure of *Predicted Share Price Dissimilarity* between the prices of months r and t, $\Delta_{PS}(p^r,p^t,q^r,q^t)$, is defined as follows:

$$(106)\ \Delta_{PS}(p^r,p^t,q^r,q^t) \equiv \Sigma_{n=1}^N [s_{t,n} - s_{r,t,n}]^2 + \Sigma_{n=1}^N [s_{r,n} - s_{t,r,n}]^2\ ; \qquad r = 1,...,T;\ t = 1,...,T.$$

The entire set of predicted share dissimilarity measures for our empirical example is a 72 by 72 element (symmetric) matrix. Table 16 below lists the first 12 rows and columns of the matrix of the bilateral measures of Predicted Share Price Dissimilarity for our empirical example.

Table 16: Month to Month Predicted Share Measures of Price Dissimilarity Using Zeros for Missing Prices

r,t	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0.1029	0.1075	0.1115	0.4470	0.5477	0.6367	0.6410	0.3713	0.1441	0.0157	0.0022
2	0.1029	0	0.0028	0.0122	0.2387	0.6201	0.6924	0.7014	0.4901	0.2498	0.1198	0.1051
3	0.1075	0.0028	0	0.0062	0.2353	0.6254	0.6967	0.7089	0.4909	0.2562	0.1261	0.1111
4	0.1115	0.0122	0.0062	0	0.2097	0.5398	0.6203	0.6285	0.4593	0.2865	0.1359	0.1073
5	0.4470	0.2387	0.2353	0.2097	0	0.0539	0.0912	0.1017	0.3485	0.2456	0.3900	0.3686
6	0.5477	0.6201	0.6254	0.5398	0.0539	0	0.0250	0.0795	0.2432	0.2248	0.3883	0.4635
7	0.6367	0.6924	0.6967	0.6203	0.0912	0.0250	0	0.0204	0.1716	0.1974	0.3854	0.5560
8	0.6410	0.7014	0.7089	0.6285	0.1017	0.0795	0.0204	0	0.1224	0.1472	0.3619	0.5584
9	0.3713	0.4901	0.4909	0.4593	0.3485	0.2432	0.1716	0.1224	0	0.0148	0.1963	0.3671
10	0.1441	0.2498	0.2562	0.2865	0.2456	0.2248	0.1974	0.1472	0.0148	0	0.0956	0.1429
11	0.0157	0.1198	0.1261	0.1359	0.3900	0.3883	0.3854	0.3619	0.1963	0.0956	0	0.0123
12	0.0022	0.1051	0.1111	0.1073	0.3686	0.4635	0.5560	0.5584	0.3671	0.1429	0.0123	0

The set of real time links which minimize the above dissimilarity measures for the first 12 observations are as follows:

11
|
1 – 2 – 3 – 4 – 5 – 6 – 7 – 8 – 9 – 10
|
12

It can be seen that the new set of bilateral links is the set of links that generates chained Fisher indexes for months 1 to 10. However, months 11 and 12 are linked directly to month 1. It can also be seen that the measures of price dissimilarity in the above Table 16 are much bigger than the corresponding measures in Table 14, which used artificial carry forward/backward prices for the missing prices. It turns out that the set of bilateral links for the first 12 months basically determines the seasonal fluctuations for the similarity linked indexes P_S^{t*} for the remainder of the sample.[45]

The Predicted Share indexes (using maximum overlap bilateral Fisher indexes as the basic building blocks) P_S^{t*} along with the other 7 indexes defined in this section are listed below in Table 17.

Table 17: Alternative Month to Month Price Indexes Using Maximum Overlap Bilateral Indexes as Building Blocks

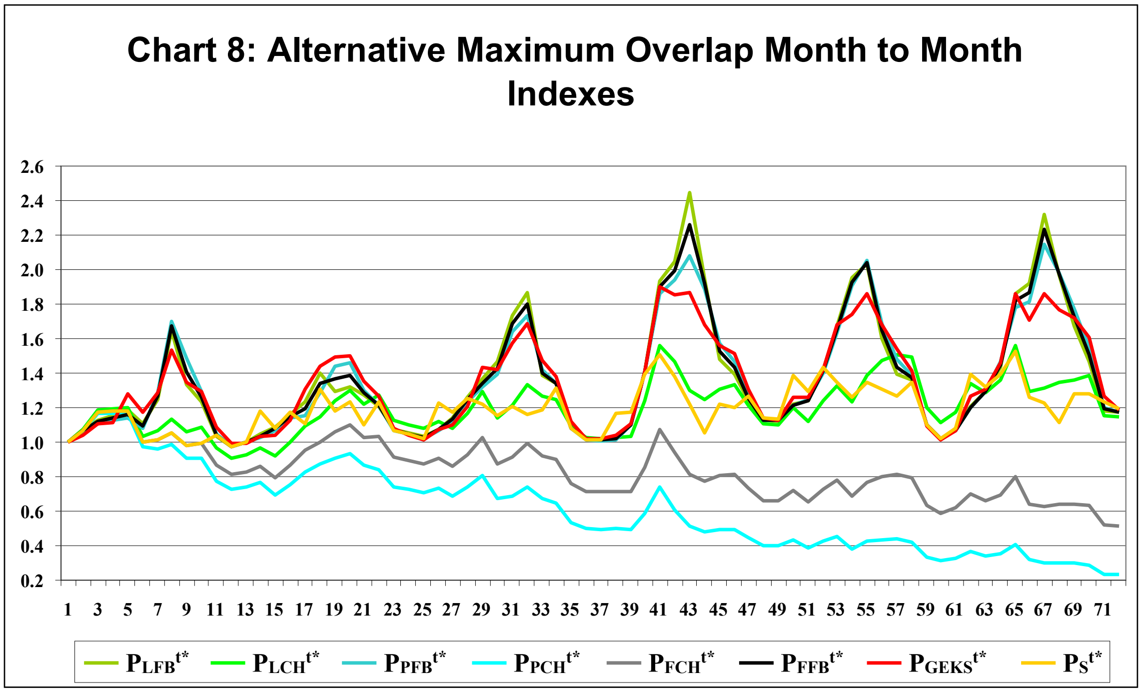
t	P_{LFB}^{t*}	P_{LCH}^{t*}	P_{PFB}^{t*}	P_{PCH}^{t*}	P_{FCH}^{t*}	P_{FFB}^{t*}	P_{GEKS}^{t*}	P_S^{t*}
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.07104	1.07104	1.06104	1.06104	1.06603	1.06603	1.03802	1.06603
3	1.12812	1.18503	1.11303	1.16798	1.17647	1.12055	1.10386	1.17647
4	1.15044	1.19078	1.12373	1.16845	1.17956	1.13701	1.11167	1.17956
5	1.18406	1.19694	1.14104	1.16942	1.18310	1.16235	1.28331	1.18310
6	1.10502	1.03417	1.07887	0.97269	1.00296	1.09186	1.17550	1.00296
7	1.24566	1.06832	1.28386	0.95860	1.01198	1.26462	1.28536	1.01198
8	1.64472	1.13041	1.69981	0.98562	1.05554	1.67204	1.53539	1.05554
9	1.33555	1.05897	1.48835	0.90641	0.97973	1.40988	1.34806	0.97973
10	1.23076	1.08596	1.29420	0.90374	0.99067	1.26208	1.29133	0.99067
11	1.03294	0.96785	1.04925	0.77360	0.86529	1.04107	1.08720	1.04107
12	0.97081	0.90818	0.98105	0.72496	0.81141	0.97592	0.99061	0.97592
13	0.99746	0.92454	0.99881	0.74299	0.82881	0.99813	0.99017	0.99684
14	1.04362	0.96387	1.02824	0.76965	0.86130	1.03590	1.03346	1.17902
15	1.08902	0.91833	1.06632	0.69057	0.79635	1.07761	1.04121	1.08056
16	1.15867	0.99822	1.13743	0.75088	0.86576	1.14800	1.12905	1.17474
17	1.23204	1.09582	1.15330	0.82753	0.95227	1.19202	1.30850	1.10498
18	1.40150	1.14776	1.28409	0.87289	1.00094	1.34151	1.44174	1.30841
19	1.29310	1.24169	1.44276	0.90848	1.06210	1.36588	1.49357	1.18142
20	1.32299	1.30180	1.45909	0.93116	1.10099	1.38937	1.50199	1.23391
21	1.27093	1.22152	1.30112	0.86403	1.02734	1.28593	1.35277	1.09986
22	1.20769	1.27442	1.21638	0.83745	1.03309	1.21203	1.26983	1.23179
23	1.07109	1.12680	1.06924	0.74250	0.91468	1.07017	1.07993	1.06906
24	1.05248	1.10170	1.04479	0.72412	0.89318	1.04863	1.04214	1.04392
25	1.03276	1.07917	1.01894	0.70949	0.87502	1.02583	1.01037	1.02270
26	1.07388	1.12065	1.06790	0.73177	0.90557	1.07089	1.07080	1.22856
27	1.14208	1.08103	1.12088	0.68492	0.86047	1.13143	1.09977	1.17215
28	1.26148	1.16653	1.23415	0.73941	0.92873	1.24774	1.20934	1.25327
29	1.36612	1.29576	1.31684	0.80866	1.02364	1.34125	1.43467	1.22223
30	1.46661	1.13765	1.39075	0.67317	0.87512	1.42818	1.42150	1.15449
31	1.73296	1.21390	1.64024	0.68759	0.91360	1.68596	1.57446	1.20526
32	1.86733	1.33342	1.73601	0.73833	0.99222	1.80047	1.68844	1.16278
33	1.38675	1.26931	1.41926	0.67018	0.92232	1.40291	1.47063	1.18929
34	1.33751	1.24621	1.33703	0.64654	0.89762	1.33727	1.38167	1.31066
35	1.08703	1.09047	1.07810	0.53089	0.76086	1.08256	1.11771	1.07810
36	1.02305	1.02489	1.01285	0.49852	0.71479	1.01793	1.01873	1.01195
37	1.01159	1.02029	1.00992	0.49537	0.71093	1.01076	1.01965	1.01076
38	1.02156	1.02791	1.01478	0.49739	0.71503	1.01816	1.03899	1.16812
39	1.10562	1.03457	1.10047	0.49269	0.71395	1.10305	1.10616	1.17108

[45] The remainder of the real time maximum overlap bilateral Fisher index links for the next 60 months are as follows: 13/12, 14/3, 15/2, 16/15, 17/5, 18/6, 19/8, 20/8, 21/9, 22/11, 23/12 and 24/23, 25/24, 26/14, 27/15, 28/4, 29/17, 30/18, 31/30, 32/19, 33/21, 34/11, 35/23, 36/23, 37/1, 38/26, 39/27, 40/28, 41/29, 42/30, 43/31, 44/32, 45/21, 46/10, 47/11, 48/25, 49/48, 50/38, 51/16, 52/51, 53/29, 54/30, 55/43, 56/20, 57/21, 58/35, 59/25, 60/59, 61/59, 62/50, 63/2, 64/40, 65/41, 66/54, 67/43, 68/44, 69/9, 70/46, 71/22 and 72/49.

40	**1.40435**	**1.24295**	**1.40366**	**0.58418**	**0.85212**	**1.40401**	**1.40571**	**1.39663**
41	**1.93372**	**1.56266**	**1.86189**	**0.73864**	**1.07436**	**1.89746**	**1.90004**	**1.50841**
42	**2.04948**	**1.46603**	**1.93935**	**0.60514**	**0.94189**	**1.99365**	**1.85555**	**1.37756**
43	**2.44964**	**1.29749**	**2.08106**	**0.51185**	**0.81494**	**2.25784**	**1.86789**	**1.22151**
44	**1.94826**	**1.24778**	**1.88870**	**0.48280**	**0.77616**	**1.91825**	**1.68235**	**1.05506**
45	**1.47755**	**1.30689**	**1.57473**	**0.49458**	**0.80396**	**1.52537**	**1.55868**	**1.22173**
46	**1.39276**	**1.33285**	**1.46928**	**0.49621**	**0.81325**	**1.43051**	**1.51589**	**1.19999**
47	**1.24213**	**1.21568**	**1.26300**	**0.44463**	**0.73520**	**1.25252**	**1.30463**	**1.26828**
48	**1.12808**	**1.10394**	**1.12696**	**0.39854**	**0.66330**	**1.12752**	**1.13747**	**1.13921**
49	**1.12212**	**1.09971**	**1.12896**	**0.39694**	**0.66070**	**1.12554**	**1.12370**	**1.13475**
50	**1.21916**	**1.20145**	**1.21377**	**0.43159**	**0.72010**	**1.21646**	**1.25767**	**1.38339**
51	**1.23969**	**1.12039**	**1.24530**	**0.38419**	**0.65608**	**1.24249**	**1.25967**	**1.29063**
52	**1.42259**	**1.24305**	**1.39905**	**0.42690**	**0.72847**	**1.41077**	**1.42183**	**1.43303**
53	**1.67018**	**1.32603**	**1.63196**	**0.45535**	**0.77705**	**1.65096**	**1.67706**	**1.34386**
54	**1.95352**	**1.23483**	**1.90595**	**0.38325**	**0.68793**	**1.92959**	**1.74172**	**1.25757**
55	**2.03052**	**1.38379**	**2.05315**	**0.42804**	**0.76962**	**2.04180**	**1.85986**	**1.34547**
56	**1.60294**	**1.47400**	**1.68914**	**0.43463**	**0.80040**	**1.64547**	**1.68088**	**1.30412**
57	**1.39502**	**1.50826**	**1.47791**	**0.43737**	**0.81220**	**1.43587**	**1.53684**	**1.26875**
58	**1.35851**	**1.49533**	**1.38700**	**0.41899**	**0.79153**	**1.37268**	**1.41013**	**1.34737**
59	**1.09904**	**1.19676**	**1.09780**	**0.33469**	**0.63288**	**1.09842**	**1.09276**	**1.09738**
60	**1.02215**	**1.11101**	**1.02087**	**0.31099**	**0.58780**	**1.02151**	**1.01005**	**1.01922**
61	**1.07410**	**1.17127**	**1.06543**	**0.32854**	**0.62033**	**1.06976**	**1.06802**	**1.07767**
62	**1.20643**	**1.33804**	**1.19934**	**0.36770**	**0.70142**	**1.20288**	**1.26692**	**1.39115**
63	**1.29331**	**1.28447**	**1.29424**	**0.33803**	**0.65893**	**1.29377**	**1.30348**	**1.32072**
64	**1.43515**	**1.35774**	**1.46351**	**0.35631**	**0.69554**	**1.44926**	**1.45723**	**1.39001**
65	**1.86123**	**1.56164**	**1.77723**	**0.40988**	**0.80006**	**1.81874**	**1.86177**	**1.52597**
66	**1.91700**	**1.29176**	**1.81516**	**0.31667**	**0.63958**	**1.86539**	**1.70997**	**1.25740**
67	**2.31683**	**1.31065**	**2.14817**	**0.30257**	**0.62973**	**2.23091**	**1.86260**	**1.22459**
68	**1.96840**	**1.34722**	**1.98236**	**0.30251**	**0.63840**	**1.97537**	**1.76427**	**1.11160**
69	**1.67463**	**1.35833**	**1.77584**	**0.30061**	**0.63901**	**1.72450**	**1.72233**	**1.27951**
70	**1.46701**	**1.38437**	**1.54608**	**0.28995**	**0.63356**	**1.50602**	**1.60516**	**1.27885**
71	**1.18124**	**1.15524**	**1.20127**	**0.23304**	**0.51886**	**1.19121**	**1.26341**	**1.23088**
72	**1.17122**	**1.14424**	**1.17533**	**0.23170**	**0.51490**	**1.17327**	**1.18952**	**1.19115**
Mean	**1.35950**	**1.19770**	**1.35160**	**0.61718**	**0.83694**	**1.35520**	**1.34680**	**1.18920**

The maximum overlap fixed base Laspeyres and Paasche indexes, $P_{LFB}{}^{t*}$ and $P_{PFB}{}^{t*}$, end up at much the same place (1.17122 and 1.17533) and have similar means (1.35950 and 1.35160). The chained Laspeyres and Paasche indexes, $P_{LCH}{}^{t*}$ and $P_{PCH}{}^{t*}$, suffer from some downward chain drift and end up far apart at 1.14424 and 0.23170 respectively. The downward chain drift problem carries over to the maximum overlap chained Fisher index, $P_{FCH}{}^{t*}$, which ends up at 0.51490. Our three best indexes from the viewpoint of controlling substitution bias and chain drift bias, $P_{FFB}{}^{t*}$, $P_{GEKS}{}^{t*}$ and $P_S{}^{t*}$, end up at 1.17327, 1.18952[46] and 1.19115 respectively. The means of the $P_{FFB}{}^{t*}$ and $P_{GEKS}{}^{t*}$ are similar at 1.3552 and 1.3468. These means are far above the mean of the similarity linked indexes $P_S{}^{t*}$ which is 1.1892. It turns out that the seasonal fluctuations in the maximum overlap fixed base Fisher indexes and the GEKS indexes are very much bigger than the seasonal fluctuations in the Predicted Share similarity linked indexes $P_S{}^{t*}$ as can be seen in the following Chart 8.

[46] From Table 15, the carry forward GEKS index ended up at 1.17327. Using maximum overlap bilateral Fisher indexes, the resulting GEKS index ended up at 1.18952. Thus the use of carry forward prices led to a downward bias of 1.62 percentage points over the 6 year sample period.



The chained Paasche and Fisher indexes suffer from a massive amount of downward chain drift. The remaining 6 indexes end up in much the same place. However, the seasonal peaks in 4 of the remaining indexes (the fixed base Laspeyres and Paasche indexes, the fixed base Fisher and the GEKS indexes) are huge. The Maximum Overlap Predicted Share similarity linked index P_S^{t*} has the best axiomatic properties (no chain drift and little or no substitution bias) and it has limited seasonal fluctuations for our empirical example so *it emerges as our best index*. From Chart 8, it can be seen that the chained Maximum Overlap Laspeyres index P_{LCH}^{t*} turns out to be fairly close to our similarity linked indexes and thus for this empirical example, it provides an adequate approximation to our preferred indexes. For our example, the downward chain drift bias in P_{LCH}^{t*} just nicely counterbalances the upward substitution bias that is inherent in the Laspeyres formula.

Chart 8 also reveals another interesting property of our empirical example. For the months of December, January and February, the three superlative indexes, P_{FFB}^{t*}, P_{GEKS}^{t*} and P_S^{t*}, and the two fixed base Laspeyres and Paasche indexes, P_{LFB}^{t*} and P_{PFB}^{t*}, all exhibit similar values. Thus these five indexes do capture the overall *trend* in the prices of the seasonal products in our example.

In the following two sections, we turn our attention to indexes that are based only on price information for strongly seasonal commodities. Section 8 looks at alternative price indexes that use the month to month carry forward prices that were used in section 6 while section 9 constructs month to month maximum overlap price indexes using only price data.

8. Month to Month Unweighted Price Indexes Using Carry Forward Prices

For many categories of consumer spending, statistical agencies will not have access to price and quantity (or expenditure) data pertaining to the category under consideration: only information on prices will be available. In this section, we will assume that carry forward prices are used as estimates for missing prices while in the subsequent section, we will consider price indexes that do not use carry forward prices.

For our empirical example, we will use the monthly price data that are listed in Table A23 in the Appendix. The price data in that table include month to month carry forward/backward prices.

As usual, define the period t price for product n as $p_{t,n}$ for t = 1,...,T and n = 1,...,N. Define the month t price vector as $p^t \equiv [p_{t,1},p_{t,2},...,p_{t,n}]$ for t = 1,...,T.

Price indexes for a category of commodities that depend only on prices are called *elementary price indexes*. The three most commonly used elementary indexes that measure the price level of month t relative to month r are the *Dutot* (1738), *Carli* (1764) and *Jevons* (1865) *indexes* defined below by (107)-(109):

$$(107)\ P_D(t/r) \equiv (1/N)\Sigma_{n=1}^{N} p_{t,n}/(1/N)\Sigma_{n=1}^{N} p_{r,n}\ ; \qquad r = 1,...,T;\ t = 1,...,T;$$
$$(108)\ P_C(t/r) \equiv (1/N)\Sigma_{n=1}^{N} p_{t,n}/p_{r,n}\ ; \qquad r = 1,...,T;\ t = 1,...,T;$$
$$(109)\ P_J(t/r) \equiv (\Pi_{n=1}^{N} p_{t,n})^{1/N}/(\Pi_{n=1}^{N} p_{r,n})^{1/N} \qquad r = 1,...,T;\ t = 1,...,T;$$
$$= (\Pi_{n=1}^{N} p_{t,n}/p_{r,n})^{1/N}\ .$$

Thus the Dutot bilateral price index between the prices of month t relative to the prices of month r is equal to the arithmetic mean of the month t prices divided by the arithmetic mean of the month r prices; the Carli bilateral price index is equal to the arithmetic mean of the month t relative to month r price ratios $p_{t,n}/p_{r,n}$ and the Jevons bilateral index is equal to the geometric mean of the month t prices divided by the geometric mean of the month r prices, which in turn is equal to the geometric mean of the month t relative to month r price ratios $p_{t,n}/p_{r,n}$.

The sequence of T *fixed base Dutot indexes* using carry forward prices, P_D^t, is $P_D(1/1)$, $P_D(2/1)$, ..., $P_D(T/1)$. The sequence of T *fixed base Carli indexes* using carry forward prices, P_{CFB}^t, is $P_C(1/1)$, $P_C(2/1)$, ..., $P_C(T/1)$ and the sequence of T *fixed base Jevons indexes* using carry forward prices, P_J^t, is $P_J(1/1)$, $P_J(2/1)$, ..., $P_J(T/1)$. We use the data listed in Table A23 in the Appendix to calculate these indexes for our Israeli data set. These indexes are listed in Table 18 below.

Define the *month to month chained Carli index using carry forward prices* for month 1 as unity:

$$(110)\ P_{CCH}^1 \equiv 1.$$

For months following month 1, the *chained Carli indexes* are calculated by cumulating the corresponding successive month to month links using definition (108); i.e., we have the following definition for P_{CCH}^t:

$$(111)\ P_{CCH}^t \equiv P_{CCH}^{t-1}P_C(t/(t-1))\ ; \qquad t = 2,3,...,T.$$

It is easy to show that the chained Dutot and Jevons indexes are equal to their fixed base counterpart indexes when there are no missing prices, as is the case in this section. This explains why we labeled the fixed base Dutot and Jevons indexes for month t as P_D^t and P_J^t instead of P_{DFB}^t and P_{JFB}^t or P_{DCH}^t and P_{JCH}^t: when there are no missing prices, $P_{DFB}^t = P_{DCH}^t \equiv P_D^t$ and $P_{JFB}^t = P_{JCH}^t \equiv P_J^t$. The chained Carli indexes P_{CCH}^t are also listed in Table 18 below.

The problem with the chained Carli indexes is that they do not satisfy the time reversal test; i.e., we have the following inequality:

(112) $P_C(2/1)P_C(1/2) \geq 1$.

The inequality in (112) will be strict unless the prices of month 1 are proportional to the prices in month 2. Thus the Carli index is subject to some upward bias whenever the base period is changed.

The problem with the Dutot index is that it is not invariant to changes in the units of measurement. This makes the use of the Dutot index problematic.[47]

We cannot apply the economic approach to index number theory in the present context since the economic approach depends on the availability of price and quantity (or expenditure) data.

From the perspective of the test or axiomatic approach to index number theory when only price data are available, the Jevons index seems to be the best choice since it satisfies the most "reasonable" tests.[48]

Table 18 below lists the Jevons, Dutot, fixed base and chained Carli indexes using carry forward prices along with our two multilateral indexes from the previous section that used bilateral maximum overlap Fisher indexes as their basic building blocks, the GEKS and Predicted Share Similarity Linked indexes, $P_{GEKS}{}^{t*}$ and $P_S{}^{t*}$.

Table 18: The Jevons, Dutot, Fixed Base and Chained Carli Indexes using Carry Forward Prices, the Maximum Overlap GEKS Index and the Maximum Overlap Similarity Linked Index

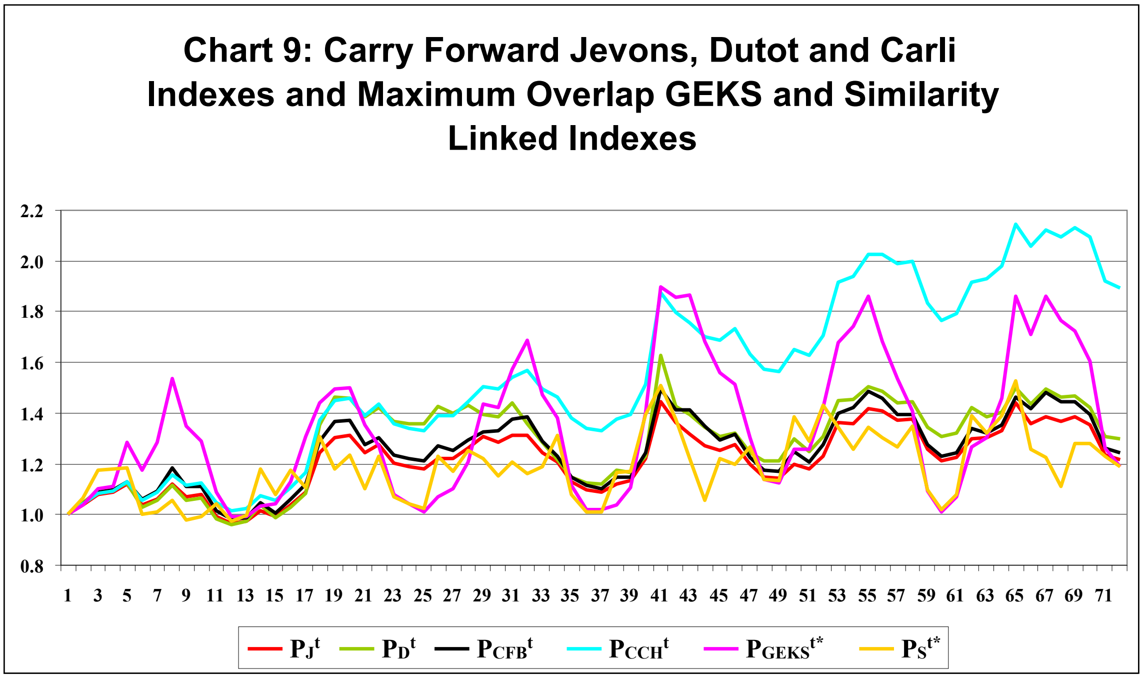
t	$P_J{}^t$	$P_D{}^t$	$P_{CFB}{}^t$	$P_{CCH}{}^t$	$P_{GEKS}{}^{t*}$	$P_S{}^{t*}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.04010	1.04700	1.04392	1.04392	1.03802	1.06603
3	1.07798	1.09168	1.08893	1.08363	1.10386	1.17647
4	1.08732	1.09744	1.09756	1.09335	1.11167	1.17956
5	1.11941	1.12322	1.12887	1.13011	1.28331	1.18310
6	1.03920	1.03101	1.06188	1.05794	1.17550	1.00296
7	1.06297	1.05679	1.09182	1.08860	1.28536	1.01198
8	1.11825	1.11627	1.18440	1.15733	1.53539	1.05554
9	1.07196	1.05739	1.11012	1.11731	1.34806	0.97973
10	1.07739	1.06336	1.10991	1.12656	1.29133	0.99067
11	0.99416	0.98341	1.01428	1.04686	1.08720	1.04107
12	0.96384	0.96114	0.98076	1.01561	0.99061	0.97592
13	0.97250	0.97198	0.98866	1.02557	0.99017	0.99684
14	1.01292	1.03900	1.04803	1.07287	1.03346	1.17902
15	0.99303	0.98812	1.00797	1.05799	1.04121	1.08056
16	1.03873	1.03056	1.06138	1.10862	1.12905	1.17474
17	1.08971	1.08152	1.12162	1.16750	1.30850	1.10498
18	1.24544	1.35291	1.29102	1.37201	1.44174	1.30841
19	1.30520	1.46275	1.36891	1.44742	1.49357	1.18142

[47] One might try to eliminate the problem of a lack of invariance of the Dutot index to changes in the units of measurement by using normalized prices; i.e., prices divided by the price of each product at the beginning of the sample period. In this case, the normalized fixed base Dutot index of prices in period t relative to prices in period 1 becomes $P_{DN}(t/1) \equiv (1/N)\Sigma_{n=1}{}^N (p_{tn}/p_{1n})/(1/N)\Sigma_{n=1}{}^N (p_{1n}) = (1/N)\Sigma_{n=1}{}^N (p_{tn}/p_{1n}) = P_{CFB}(t/1)$. Thus the normalized fixed base Dutot index becomes the fixed base Carli index.

[48] For materials on the test approach to bilateral index number theory when only price information is available, see Eichhorn (1978; 152-160), Dalén (1992) and Diewert (1995; 5-17) (2021a).

20	**1.31250**	**1.46029**	**1.36965**	**1.45718**	**1.50199**	**1.23391**
21	**1.24244**	**1.38497**	**1.27474**	**1.39094**	**1.35277**	**1.09986**
22	**1.27503**	**1.42091**	**1.30348**	**1.43432**	**1.26983**	**1.23179**
23	**1.20426**	**1.36614**	**1.23569**	**1.35761**	**1.07993**	**1.06906**
24	**1.18828**	**1.36016**	**1.22022**	**1.34060**	**1.04214**	**1.04392**
25	**1.17994**	**1.35881**	**1.21315**	**1.33167**	**1.01037**	**1.02270**
26	**1.22319**	**1.42860**	**1.27256**	**1.39023**	**1.07080**	**1.22856**
27	**1.21864**	**1.39737**	**1.25123**	**1.39093**	**1.09977**	**1.17215**
28	**1.26329**	**1.42957**	**1.29381**	**1.44402**	**1.20934**	**1.25327**
29	**1.30577**	**1.39595**	**1.32502**	**1.50247**	**1.43467**	**1.22223**
30	**1.28390**	**1.38736**	**1.33158**	**1.49770**	**1.42150**	**1.15449**
31	**1.31226**	**1.43832**	**1.37750**	**1.54156**	**1.57446**	**1.20526**
32	**1.31447**	**1.35986**	**1.38423**	**1.56831**	**1.68844**	**1.16278**
33	**1.24460**	**1.28656**	**1.28806**	**1.49633**	**1.47063**	**1.18929**
34	**1.20937**	**1.21542**	**1.23065**	**1.46471**	**1.38167**	**1.31066**
35	**1.13018**	**1.14885**	**1.14550**	**1.38135**	**1.11771**	**1.07810**
36	**1.09730**	**1.12464**	**1.11395**	**1.34192**	**1.01873**	**1.01195**
37	**1.08754**	**1.12135**	**1.10424**	**1.33070**	**1.01965**	**1.01076**
38	**1.11967**	**1.17440**	**1.14982**	**1.37428**	**1.03899**	**1.16812**
39	**1.13238**	**1.16476**	**1.14976**	**1.39389**	**1.10616**	**1.17108**
40	**1.21987**	**1.22910**	**1.24218**	**1.51298**	**1.40571**	**1.39663**
41	**1.44696**	**1.62946**	**1.49635**	**1.87338**	**1.90004**	**1.50841**
42	**1.36353**	**1.42561**	**1.41370**	**1.79906**	**1.85555**	**1.37756**
43	**1.31918**	**1.39333**	**1.41085**	**1.75785**	**1.86789**	**1.22151**
44	**1.27183**	**1.34559**	**1.35040**	**1.69959**	**1.68235**	**1.05506**
45	**1.25115**	**1.30950**	**1.29494**	**1.68977**	**1.55868**	**1.22173**
46	**1.27617**	**1.32078**	**1.31579**	**1.73272**	**1.51589**	**1.19999**
47	**1.19680**	**1.24382**	**1.22543**	**1.63053**	**1.30463**	**1.26828**
48	**1.14969**	**1.21288**	**1.17680**	**1.57115**	**1.13747**	**1.13921**
49	**1.14226**	**1.21139**	**1.16845**	**1.56196**	**1.12370**	**1.13475**
50	**1.19976**	**1.29769**	**1.24905**	**1.65044**	**1.25767**	**1.38339**
51	**1.17818**	**1.24643**	**1.20921**	**1.62988**	**1.25967**	**1.29063**
52	**1.23094**	**1.30561**	**1.27799**	**1.70762**	**1.42183**	**1.43303**
53	**1.36494**	**1.44893**	**1.39965**	**1.91801**	**1.67706**	**1.34386**
54	**1.35981**	**1.45236**	**1.42396**	**1.93863**	**1.74172**	**1.25757**
55	**1.41772**	**1.50504**	**1.48798**	**2.02778**	**1.85986**	**1.34547**
56	**1.40965**	**1.48435**	**1.45820**	**2.02692**	**1.68088**	**1.30412**
57	**1.37418**	**1.43989**	**1.39525**	**1.99136**	**1.53684**	**1.26875**
58	**1.37849**	**1.44512**	**1.39683**	**2.00074**	**1.41013**	**1.34737**
59	**1.25855**	**1.34671**	**1.27672**	**1.83624**	**1.09276**	**1.09738**
60	**1.20970**	**1.30703**	**1.22881**	**1.76674**	**1.01005**	**1.01922**
61	**1.22550**	**1.32041**	**1.24234**	**1.79129**	**1.06802**	**1.07767**
62	**1.30044**	**1.42390**	**1.33984**	**1.91774**	**1.26692**	**1.39115**
63	**1.30188**	**1.38773**	**1.31950**	**1.93130**	**1.30348**	**1.32072**
64	**1.33256**	**1.40522**	**1.35153**	**1.98181**	**1.45723**	**1.39001**
65	**1.43988**	**1.49959**	**1.46360**	**2.14730**	**1.86177**	**1.52597**
66	**1.35995**	**1.43690**	**1.41659**	**2.05853**	**1.70997**	**1.25740**
67	**1.38749**	**1.49361**	**1.48206**	**2.12086**	**1.86260**	**1.22459**
68	**1.36725**	**1.46387**	**1.44619**	**2.09556**	**1.76427**	**1.11160**
69	**1.38498**	**1.47007**	**1.44701**	**2.13057**	**1.72233**	**1.27951**
70	**1.35354**	**1.42330**	**1.39351**	**2.09306**	**1.60516**	**1.27885**
71	**1.23599**	**1.30957**	**1.26408**	**1.92316**	**1.26341**	**1.23088**
72	**1.21520**	**1.29702**	**1.24216**	**1.89371**	**1.18952**	**1.19115**
Mean	**1.21640**	**1.28490**	**1.25200**	**1.54240**	**1.34680**	**1.18920**

Our "best" index from the previous section, the maximum overlap Predicted Share index, P_S^{t*}, finished up at 1.19115 which is close to where the maximum overlap GEKS index, P_{GEKS}^{t*}, finished at 1.18952. We preferred P_S^{t*} over P_{GEKS}^{t*} because the similarity linked index had better axiomatic properties and the seasonal fluctuations in P_{GEKS}^{t*} were very large. The carry forward Jevons index P_J^t performed pretty well compared to P_S^{t*}: P_J^t ended up at 1.21520 (compared to 1.19115 for P_S^{72*}) and the mean of the P_J^t was 1.2164 compared to the mean of the P_S^{t*}, which was 1.1892. The next best performing unweighted index is the fixed base Carli index which finished up at 1.24216 (mean was 1.2420), which is 5.1 percentage points above $P_S^{72*} = 1.19115$. The Dutot index ended up at 1.29702, which is 10.6 percentage points above P_S^{72*}. Finally, the chained Carli index, P_{CCH}^t, exhibited tremendous upward chain drift, ending up at 1.89952, which is 70.8 percentage points above P_S^{72*}. Chart 9 below plots these indexes.



It can be seen from Chart 9, the Jevons index P_J^t approximates our "best" index P_S^{t*} fairly well; the two indexes end up in much the same place with P_S^{t*} and the indexes are always close to each other for the months of December, January and February. For mid year months, P_S^{t*} is generally below P_J^t. The Carli fixed base and Dutot indexes are in general close to each other and tend to lie above their Jevons index counterparts. The seasonal fluctuations in the GEKS and chained Carli indexes are very large indeed. Finally, the upward chain drift in the chained Carli index is evident by looking at Chart 9.

The Jevons index that is listed in Table 18 uses carry forward prices. In previous sections, we have seen that the use of carry forward prices leads to a downward bias for our empirical example as compared to indexes which do not use carry forward prices. In the following section, we will compute additional elementary indexes that do not use quantity or expenditure weights but instead of using carry forward prices, we will use maximum overlap unweighted bilateral indexes.

9. Month to Month Unweighted Price Indexes Using Maximum Overlap Bilateral Indexes

In this section, for our empirical example, we again use the monthly price data that are listed in Table A23 in the Appendix. However, the carry forward/backward prices that are listed in italics in Table A23 are set equal to 0 in this section.

The new period t price for product n (that is equal to 0 if the product is not available) is defined as $p_{t,n}$ for t = 1,...,T and n = 1,...,N. Define the month t price vector as $p^t \equiv [p_{t,1},p_{t,2},...,p_{t,n}]$ for t = 1,...,T. As usual, the set of prices n of products that are purchased in month t is defined as S(t) for t = 1,...,T. The number of products that are purchased in month t is N(t) ≤ N. The set of products that are purchased in both months r and t is the intersection set $S(r)\cap S(t)$ and the number of matched products that are purchased in both months r and t is N(r,t).

The bilateral *Maximum Overlap Jevons, Dutot and Carli indexes* that measure the level of prices in month t relative to the prices in month r, $P_J^*(t/r)$, $P_D^*(t/r)$ and $P_C^*(t/r)$ are defined as follows:

$$(113)\ P_J^*(t/r) \equiv [\Pi_{n\in S(r)\cap S(t)}\ (p_{t,n}/p_{r,n})]^{1/N(r,t)}\ ; \qquad r = 1,...,T;\ t = 1,...,T;$$

$$(114)\ P_D^*(t/r) \equiv \Sigma_{\,n\in S(r)\cap S(t)}\ (p_{t,n}/N(r,t))/\Sigma_{\,n\in S(r)\cap S(t)}\ (p_{r,n}/N(r,t))$$
$$= \Sigma_{\,n\in S(r)\cap S(t)}\ p_{t,n}/\Sigma_{\,n\in S(r)\cap S(t)}\ p_{r,n}\ ; \qquad r = 1,...,T;\ t = 1,...,T;$$

$$(115)\ P_C^*(t/r) \equiv [1/N(r,t)]\ \Sigma_{\,n\in S(r)\cap S(t)}\ (p_{t,n}/p_{r,n})\ ; \qquad r = 1,...,T;\ t = 1,...,T.$$

The maximum overlap Jevons index $P_J^*(t/r)$ is equal to the geometric mean of the price ratios $p_{t,n}/p_{r,n}$ of the products that are present in both months r and t. The maximum overlap Dutot index $P_D^*(t/r)$ is equal to the arithmetic mean of the month t prices $p_{t,n}$ divided by the arithmetic mean of the month r prices $p_{r,n}$ where both averages include only the products that are present in both months r and t. The maximum overlap Carli index $P_C^*(t/r)$ is equal to the arithmetic average of the price ratios $p_{t,n}/p_{r,n}$ of the products that are present in both months r and t.

The sequence of monthly maximum overlap *fixed base Jevons indexes*, $P_{JFB}{}^{t*}$, is $P_J^*(2/1)$, $P_J^*(2/1)$, ..., $P_J^*(T/1)$. The sequence of maximum overlap monthly *fixed base Dutot indexes*, $P_{DFB}{}^{t*}$, is $P_D^*(1/1)$, $P_D^*(2/1)$, ..., $P_D^*(T/1)$. Finally, the sequence of maximum overlap monthly *fixed base Carli indexes*, $P_{CFB}{}^{t*}$, is $P_C^*(1/1)$, $P_C^*(2/1)$, ..., $P_C^*(T/1)$. We use the data listed in Table A23 in the Appendix (with the carry/forward and backward prices replaced by zeros) to calculate these indexes for our Israeli data set. These indexes are listed in Table 19 below.

The maximum overlap bilateral Jevons, Dutot and Carli indexes, $P_J^*(t/r)$, $P_D^*(t/r)$ and $P_C^*(t/r)$, defined by (113)-(115) are used to define the *Maximum Overlap Jevons, Dutot and Carli chained indexes*, $P_{JCH}{}^{t*}$, $P_{DCH}{}^{t*}$ and $P_{CCH}{}^{t*}$, as follows:

$$(116)\ P_{JCH}{}^{1*} \equiv 1;\ P_{DCH}{}^{1*} \equiv 1;\ P_{CCH}{}^{1*} \equiv 1.$$

$$(117)\ P_{JCH}{}^{t*} \equiv P_{JCH}{}^{t-1*}P_J^*(t/[t-1])\ ; \qquad t = 2,3,...,T;$$

$$(118)\ P_{DCH}{}^{t*} \equiv P_{DCH}{}^{t-1*}P_D^*(t/[t-1])\ ; \qquad t = 2,3,...,T;$$

$$(119)\ P_{CCH}{}^{t*} \equiv P_{CCH}{}^{t-1*}P_C^*(t/[t-1])\ ; \qquad t = 2,3,...,T.$$

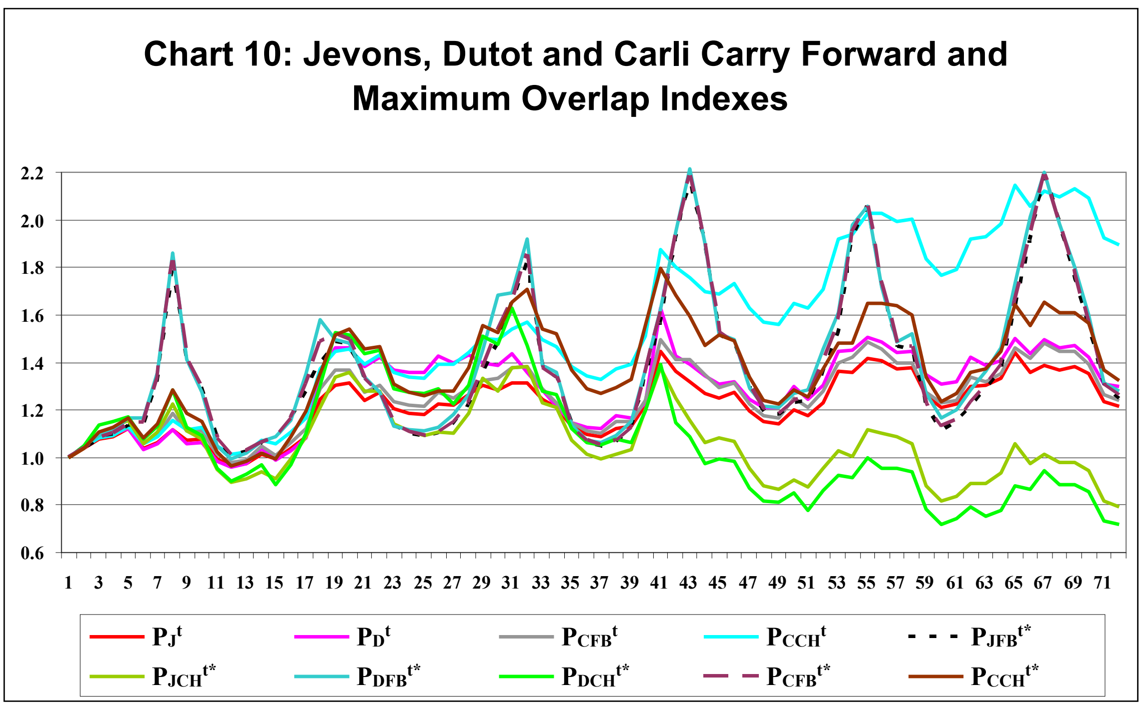
The maximum overlap Jevons and Dutot indexes are not necessarily equal to the corresponding fixed base Jevons and Dutot indexes as was the case in the previous section when carry forward prices were used as imputations for the missing prices. Thus in general, $P_{JCH}{}^{t*} \neq P_{JFB}{}^{t*}$ and $P_{DCH}{}^{t*} \neq P_{DFB}{}^{t*}$. The six elementary indexes using bilateral maximum overlap price indexes as basic building blocks, $P_{JFB}{}^{t*}$, $P_{JCH}{}^{t*}$, $P_{DFB}{}^{t*}$, $P_{DCH}{}^{t*}$, $P_{CFB}{}^{t*}$and $P_{CCH}{}^{t*}$, are listed in Table 19 below along with the four elementary indexes that used carry forward prices from the previous section, $P_J{}^t$, $P_D{}^t$, $P_{CFB}{}^t$ and $P_{CCH}{}^t$ for comparison purposes.

Table 19: The Jevons, Dutot, Fixed Base and Chained Carli Indexes using Carry Forward Prices and the Maximum Overlap Fixed Base and Chained Jevons, Dutot and Carli Indexes

t	P_J^t	P_D^t	P_{CFB}^t	P_{CCH}^t	P_{JFB}^{t*}	P_{JCH}^{t*}	P_{DFB}^{t*}	P_{DCH}^{t*}	P_{CFB}^{t*}	P_{CCH}^{t*}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.0401	1.0470	1.0439	1.0439	1.0381	1.0381	1.0484	1.0484	1.0401	1.0401
3	1.0780	1.0917	1.0889	1.0836	1.0816	1.1052	1.0944	1.1386	1.0847	1.1093
4	1.0873	1.0974	1.0976	1.0934	1.0986	1.1244	1.1131	1.1522	1.1024	1.1292
5	1.1194	1.1232	1.1289	1.1301	1.1350	1.1561	1.1655	1.1699	1.1431	1.1673
6	1.0392	1.0310	1.0619	1.0579	1.1352	1.0597	1.1666	1.0766	1.1458	1.0806
7	1.0630	1.0568	1.0918	1.0886	1.3194	1.1052	1.3589	1.1298	1.3359	1.1397
8	1.1183	1.1163	1.1844	1.1573	1.7883	1.2231	1.8620	1.2828	1.8289	1.2836
9	1.0720	1.0574	1.1101	1.1173	1.4071	1.1117	1.4128	1.1261	1.4194	1.1840
10	1.0774	1.0634	1.1099	1.1266	1.2798	1.0769	1.2758	1.0937	1.2862	1.1509
11	0.9942	0.9834	1.0143	1.0469	1.0719	0.9503	1.0493	0.9548	1.0811	1.0242
12	0.9638	0.9611	0.9808	1.0156	1.0075	0.8932	0.9919	0.9025	1.0141	0.9631
13	0.9725	0.9720	0.9887	1.0256	1.0257	0.9094	1.0199	0.9280	1.0299	0.9819
14	1.0129	1.0390	1.0480	1.0729	1.0609	0.9406	1.0653	0.9693	1.0651	1.0169
15	0.9930	0.9881	1.0080	1.0580	1.0826	0.9085	1.0879	0.8853	1.0859	0.9922
16	1.0387	1.0306	1.0614	1.1086	1.1603	0.9941	1.1579	0.9690	1.1662	1.0872
17	1.0897	1.0815	1.1216	1.1675	1.2780	1.0851	1.3548	1.1026	1.3106	1.1959
18	1.2454	1.3529	1.2910	1.3720	1.4017	1.2113	1.5774	1.3043	1.4876	1.3538
19	1.3052	1.4628	1.3689	1.4474	1.4860	1.3386	1.4979	1.5260	1.5202	1.5143
20	1.3125	1.4603	1.3697	1.4572	1.4774	1.3561	1.4807	1.5167	1.4979	1.5381
21	1.2424	1.3850	1.2747	1.3909	1.3293	1.2780	1.3405	1.4379	1.3356	1.4550
22	1.2750	1.4209	1.3035	1.4343	1.2698	1.2779	1.2723	1.4526	1.2748	1.4651
23	1.2043	1.3661	1.2357	1.3576	1.1328	1.1399	1.1311	1.2913	1.1392	1.3084
24	1.1883	1.3602	1.2202	1.3406	1.1029	1.1099	1.1156	1.2737	1.1083	1.2756
25	1.1799	1.3588	1.2132	1.3317	1.0875	1.0944	1.1122	1.2697	1.0942	1.2586
26	1.2232	1.4286	1.2726	1.3902	1.1018	1.1087	1.1285	1.2884	1.1057	1.2774
27	1.2186	1.3974	1.2512	1.3909	1.1448	1.1015	1.1806	1.2241	1.1500	1.2786
28	1.2633	1.4296	1.2938	1.4440	1.2219	1.1837	1.2630	1.3027	1.2287	1.3762
29	1.3058	1.3960	1.3250	1.5025	1.3529	1.3327	1.4509	1.5089	1.3740	1.5572
30	1.2839	1.3874	1.3316	1.4977	1.4822	1.2796	1.6819	1.4740	1.5345	1.5255
31	1.3123	1.4383	1.3775	1.5416	1.6476	1.3771	1.6923	1.6285	1.6602	1.6558
32	1.3145	1.3599	1.3842	1.5683	1.8228	1.3811	1.9192	1.4708	1.8547	1.7061
33	1.2446	1.2866	1.2881	1.4963	1.3730	1.2301	1.3871	1.2818	1.3770	1.5425
34	1.2094	1.2154	1.2307	1.4647	1.3381	1.2101	1.3565	1.2633	1.3402	1.5208
35	1.1302	1.1489	1.1455	1.3814	1.1296	1.0749	1.1349	1.1227	1.1336	1.3694
36	1.0973	1.1246	1.1140	1.3419	1.0649	1.0133	1.0725	1.0609	1.0705	1.2912
37	1.0875	1.1214	1.1042	1.3307	1.0460	0.9953	1.0640	1.0525	1.0511	1.2696
38	1.1197	1.1744	1.1498	1.3743	1.0657	1.0140	1.0912	1.0794	1.0703	1.2948
39	1.1324	1.1648	1.1498	1.3939	1.1240	1.0342	1.1538	1.0620	1.1276	1.3271
40	1.2199	1.2291	1.2422	1.5130	1.3363	1.2002	1.3684	1.1969	1.3459	1.5539
41	1.4470	1.6295	1.4964	1.8734	1.5631	1.3763	1.6218	1.3930	1.6052	1.7942
42	1.3635	1.4256	1.4137	1.7991	1.9200	1.2509	1.9490	1.1476	1.9374	1.6841
43	1.3192	1.3933	1.4109	1.7579	2.1559	1.1571	2.2132	1.0877	2.1874	1.5928
44	1.2718	1.3456	1.3504	1.6996	1.8955	1.0625	1.9020	0.9740	1.8969	1.4696
45	1.2512	1.3095	1.2949	1.6898	1.5229	1.0827	1.5208	0.9938	1.5291	1.5144
46	1.2762	1.3208	1.3158	1.7327	1.4739	1.0656	1.4939	0.9823	1.4839	1.4932
47	1.1968	1.2438	1.2254	1.6305	1.2947	0.9523	1.2935	0.8687	1.3019	1.3391
48	1.1497	1.2129	1.1768	1.5712	1.1947	0.8788	1.2137	0.8152	1.2046	1.2416
49	1.1423	1.2114	1.1685	1.5620	1.1794	0.8675	1.2099	0.8126	1.1879	1.2271
50	1.1998	1.2977	1.2491	1.6504	1.2322	0.9063	1.2685	0.8519	1.2416	1.2848
51	1.1782	1.2464	1.2092	1.6299	1.2356	0.8740	1.2854	0.7784	1.2524	1.2528
52	1.2309	1.3056	1.2780	1.7076	1.3595	0.9541	1.4546	0.8633	1.4020	1.3723
53	1.3649	1.4489	1.3997	1.9180	1.5225	1.0301	1.6111	0.9237	1.5652	1.4824

54	**1.3598**	**1.4524**	**1.4240**	**1.9386**	**1.9247**	**1.0039**	**1.9786**	**0.9129**	**1.9472**	**1.4795**
55	**1.4177**	**1.5050**	**1.4880**	**2.0278**	**2.0574**	**1.1150**	**2.0550**	**0.9991**	**2.0577**	**1.6493**
56	**1.4097**	**1.4844**	**1.4582**	**2.0269**	**1.7187**	**1.1002**	**1.7154**	**0.9544**	**1.7277**	**1.6476**
57	**1.3742**	**1.4399**	**1.3953**	**1.9914**	**1.4657**	**1.0867**	**1.4854**	**0.9566**	**1.4751**	**1.6370**
58	**1.3785**	**1.4451**	**1.3968**	**2.0007**	**1.4518**	**1.0590**	**1.5221**	**0.9389**	**1.4673**	**1.5992**
59	**1.2586**	**1.3467**	**1.2767**	**1.8362**	**1.2102**	**0.8828**	**1.2683**	**0.7824**	**1.2271**	**1.3362**
60	**1.2097**	**1.3070**	**1.2288**	**1.7667**	**1.1181**	**0.8156**	**1.1659**	**0.7192**	**1.1313**	**1.2351**
61	**1.2255**	**1.3204**	**1.2423**	**1.7913**	**1.1474**	**0.8370**	**1.2004**	**0.7405**	**1.1584**	**1.2694**
62	**1.3004**	**1.4239**	**1.3398**	**1.9177**	**1.2190**	**0.8892**	**1.2850**	**0.7927**	**1.2338**	**1.3574**
63	**1.3019**	**1.3877**	**1.3195**	**1.9313**	**1.2905**	**0.8909**	**1.3652**	**0.7523**	**1.3069**	**1.3742**
64	**1.3326**	**1.4052**	**1.3515**	**1.9818**	**1.3674**	**0.9334**	**1.4596**	**0.7753**	**1.3943**	**1.4460**
65	**1.4399**	**1.4996**	**1.4636**	**2.1473**	**1.6272**	**1.0597**	**1.7302**	**0.8827**	**1.6586**	**1.6431**
66	**1.3600**	**1.4369**	**1.4166**	**2.0585**	**1.9131**	**0.9761**	**2.0145**	**0.8679**	**1.9373**	**1.5561**
67	**1.3875**	**1.4936**	**1.4821**	**2.1209**	**2.1811**	**1.0150**	**2.2012**	**0.9464**	**2.1886**	**1.6537**
68	**1.3673**	**1.4639**	**1.4462**	**2.0956**	**1.9852**	**0.9808**	**1.9837**	**0.8861**	**1.9853**	**1.6077**
69	**1.3850**	**1.4701**	**1.4470**	**2.1306**	**1.7458**	**0.9781**	**1.8014**	**0.8853**	**1.7684**	**1.6100**
70	**1.3535**	**1.4233**	**1.3935**	**2.0931**	**1.5445**	**0.9430**	**1.5974**	**0.8549**	**1.5606**	**1.5638**
71	**1.2360**	**1.3096**	**1.2641**	**1.9232**	**1.2941**	**0.8187**	**1.3124**	**0.7349**	**1.3074**	**1.3663**
72	**1.2152**	**1.2970**	**1.2422**	**1.8937**	**1.2509**	**0.7914**	**1.2800**	**0.7168**	**1.2636**	**1.3245**
Mean	**1.2164**	**1.2849**	**1.2520**	**1.5424**	**1.3690**	**1.0647**	**1.4049**	**1.0640**	**1.3835**	**1.3662**

The 10 indexes listed in Table 19 are plotted on Chart 10 below.



The four chained indexes all seem to suffer from some form of chain drift: the maximum overlap chained Carli P_{CCH}^{t*} ends up too high at 1.3245 while its carry forward chained Carli index P_{CCH}^t ends up much too high at 1.8937. The chained maximum overlap Jevons and Dutot, P_{JCH}^{t*} and P_{DCH}^{t*}, suffer from severe downward chain drift and end up at 0.7914 and 0.7168 respectively. The carry forward Dutot index P_D^t ended up at 1.2970 and its maximum overlap fixed base counterpart P_{DFB}^{t*} ended up at 1.2800. Our "best" index using price and expenditure information was the maximum overlap similarity linked index P_S^{t*} which ended up at 1.1911. Thus the Dutot indexes $P_D{}^t$ and $P_{DFB}{}^{t*}$ have a considerable amount of upward bias relative to our preferred index. In general, the fixed base maximum overlap Jevons, Dutot and Carli indexes, $P_{JFB}{}^{t*}$, $P_{DFB}{}^{t*}$ and $P_{CFB}{}^{t*}$, are fairly close to each other but they end up at 1.2509, 1.2800 and 1.2636 respectively which is well above where the similarity linked maximum overlap index ended (1.1911). Also $P_{JFB}{}^{t*}$, $P_{DFB}{}^{t*}$ and $P_{CFB}{}^{t*}$ have large seasonal fluctuations relative to $P_S{}^{t*}$. These three maximum overlap fixed base indexes cannot be readily distinguished from each other on Chart 10. The index which provides the closest approximation to $P_S{}^{t*}$ is the Jevons index $P_J{}^t$, which uses carry forward prices.

However, as we have seen in previous sections, the use of carry forward prices can lead to significant bias as compared to the same index which uses maximum overlap indexes. From Table 19, the mean of the fixed base Jevons indexes using carry forward prices (the $P_J{}^t$) is 1.2164 while the mean of the fixed base maximum overlap indexes $P_{JFB}{}^{t*}$ is 1.3690. Thus on average, the downward bias in the use of the carry forward indexes using the Jevons formula is 1.4049 − 1.2849 or 15.26 percentage points. Similarly the downward bias in the use of carry forward prices using fixed base Dutot indexes is 1.4049 − 1.2849 or 15.29 percentage points and the downward bias in the use of carry forward prices using fixed base Carli indexes is 1.3835 − 1.2520 or 13.15 percentage points. *Thus the use of carry forward prices for elementary indexes in situations where there is general inflation cannot be recommended due to the potentially large downward bias that the use of carry forward prices can generate*.

Instead of using maximum overlap bilateral Jevons, Dutot and Carli indexes as basic inputs into fixed base and chained indexes of prices (without quantity or expenditure weights), it is possible to use multilateral methods to form elementary indexes. We conclude this section by considering two such multilateral methods that just use price information for many period: the time product dummy method and a similarity based linking method.

The *time product dummy method* assumes that the price of product n in month t, $p_{t,n}$, is approximately equal to the product of two factors: a time factor $\pi_t > 0$ that represents the *price level in month t* and a product factor, $\alpha_n > 0$ that represents the utility of product n relative to all products in scope. It is convenient to take logarithms of both sides of the approximate equations $p_{t,n} \approx \pi_t\alpha_n$ in order to obtain the approximate equations $\ln p_{t,n} \approx \ln\pi_t + \ln\alpha_n = \rho_t + \beta_n$ where $\rho_t \equiv \ln\pi_t$ and $\beta_n \equiv \ln\alpha_n$. Estimates $\rho_t{}^*$ and $\beta_n{}^*$ for the parameters ρ_t and β_n can be obtained by solving the following least squares minimization problem:

(120) $\min_{\rho,\beta} \{\Sigma_{t=1}{}^T \Sigma_{n\in S(t)} [\ln p_{t,n} - \rho_t - \beta_n]^2\}$

where we set $\rho_1 \equiv 0$ in order to prevent multicollinearity problems. Denote the solution to (120) by $\rho_t{}^*$ for t = 2,3,...,T and $\beta_n{}^*$ for n = 1,...,N. Define $\rho_1{}^* \equiv 0$ and define $\pi_t{}^* \equiv \exp[\rho_t{}^*]$ for t = 1,...,T. The *Time Product Dummy index* for month t, $P_{TPD}{}^t$, is defined to be $\pi_t{}^*$; i.e., we have $P_{TPD}{}^t \equiv \pi_t{}^*$ for t = 1,...,T.[49] If there are no missing observations so that all N products are present in all N periods, then the Time Product Dummy price indexes are equal to the fixed base (and chained) Jevons index $P_J(t/1) = P_J{}^t$.[50] Thus the Time Product Dummy index $P_{TPD}{}^t$ is a natural generalization

[49] In the statistics literature, this model is known as the fixed effects model. In the economics literature, the method is due to Court (1939; 109-111) in the hedonic regression context and to Summers (1973) in the international comparison context where it is known as the Country Product Dummy regression model. See Diewert (2021c) for more on the history of this multilateral method and its interpretation from the perspective of the economic approach to index number theory.
[50] This result is a special case of a more general result obtained by Triplett and McDonald (1977; 150). See also Diewert (2021c; 51).

of the Jevons index to the case of missing observations. This standard Time Product Dummy index P_{TPD}^t is listed in Table 22 and plotted in Chart 11 below.

It is of interest to calculate year over year maximum overlap fixed base and chained Jevons indexes for each month. Denote the sequence of year over year fixed base and chained maximum overlap Jevons indexes for month m and year y as P_{JFm}^{y*} and P_{JCm}^{y*} respectively for m = 1,...,M and y = 1,...,Y. These month over month Jevons indexes are listed in Tables 20 and 21 for our empirical example.

Table 20: Year over Year Monthly Maximum Overlap Fixed Base Jevons Indexes

y	P_{JF1}^{y*}	P_{JF2}^{y*}	P_{JF3}^{y*}	P_{JF4}^{y*}	P_{JF5}^{y*}	P_{JF6}^{y*}	P_{JF7}^{y*}	P_{JF8}^{y*}	P_{JF9}^{y*}	P_{JF10}^{y*}	P_{JF11}^{y*}	P_{JF12}^{y*}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.0257	1.0761	0.9763	1.0341	1.0051	1.3229	1.3156	1.1127	1.0596	1.0734	1.0568	1.0947
3	1.0875	1.1228	1.0478	1.0916	1.1310	1.2719	1.2127	1.1121	1.1080	1.1008	1.0698	1.0570
4	1.0460	1.0815	1.0361	1.1838	1.2999	1.4799	1.2876	1.0803	1.1763	1.1953	1.2225	1.1859
5	1.1794	1.2487	1.1236	1.2055	1.2587	1.4438	1.3801	1.2180	1.1986	1.1544	1.1290	1.1097
6	1.1474	1.2449	1.1716	1.1927	1.3593	1.4416	1.3280	1.1518	1.2860	1.2645	1.2320	1.2416

Table 21: Year over Year Monthly Maximum Overlap Chained Jevons Indexes

y	P_{JC1}^{y*}	P_{JC2}^{y*}	P_{JC3}^{y*}	P_{JC4}^{y*}	P_{JC5}^{y*}	P_{JC6}^{y*}	P_{JC7}^{y*}	P_{JC8}^{y*}	P_{JC9}^{y*}	P_{JC10}^{y*}	P_{JC11}^{y*}	P_{JC12}^{y*}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.0257	1.0761	0.9763	1.0341	1.0051	1.3229	1.3156	1.1127	1.0596	1.0734	1.0568	1.0947
3	1.0875	1.1228	1.0478	1.0916	1.0909	1.2106	1.2166	1.1200	1.1080	1.1008	1.0539	1.0570
4	1.0460	1.0815	1.0361	1.1838	1.2919	1.3731	1.2876	1.0803	1.1763	1.1842	1.2079	1.1859
5	1.1794	1.2487	1.1218	1.2055	1.1940	1.3397	1.3801	1.2180	1.1986	1.1804	1.1290	1.1097
6	1.1474	1.2449	1.1563	1.1927	1.2893	1.3376	1.3280	1.1518	1.2860	1.2414	1.2073	1.2416

It can be seen that, for the most part, the fixed base Jevons indexes in Table 20 approximated their chained counterparts in Table 21 fairly well. For the months m where the list of available products is the same for all years, the fixed base and chained maximum overlap indexes for those months will be the same; i.e., we have $P_{JFm}^{y*} = P_{JCm}^{y*}$ for y = 1,...,6 for months m where the available products are always the same year over year.

A possible disadvantage of using the Time Product Dummy indexes P_{TPD}^t is that every month when there is a new observation, the indexes have to be recomputed and there is the problem of linking the new index for the latest month with the prior indexes. A possible solution to this problem is the following one. (i) Compute the Time Product Dummy indexes for a historical data set that consists of 12 consecutive months. Call the resulting indexes P_{TPD}^t for m = 1,...,12. (ii) Set the *Mixed TPD and Jevons index*, P_{TPDJ}^t, for the first 12 months equal to the corresponding Time Product Dummy indexes so that $P_{TPDJ}^t = P_{TPD}^t$ for t = 1,...,12. (ii) For subsequent months, use the year over year fixed base maximum overlap Jevons indexes P_{JFm}^{y*} to link month m in year y ≥ 2 to P_{TPDJ}^m. Thus for our empirical example, for year y = 2, we have $P_{TPDJ}^{12+m} = P_{TPDJ}^m \times P_{JFm}^{2*}$ for m = 1,...,12; for year y = 3, we have $P_{TPDJ}^{24+m} = P_{TPDJ}^m \times P_{JFm}^{3*}$ for m = 1,...,12; ... and for year y = 6, we have $P_{TPDJ}^{60+m} = P_{TPDJ}^m \times P_{JFm}^{6*}$ for m = 1,...,12. The reason for using the fixed base monthly maximum overlap year over year Jevons indexes listed in Table 20 instead of the chained indexes listed in Table 21 is that the resulting Mixed TPD and Jevons indexes, P_{TPDJ}^t satisfy Walsh's multiperiod identity test; i.e., if prices in months r and t are the same, then $P_{TPDJ}^r = P_{TPDJ}^t$.[51] If an index satisfies this test, then it is free from chain drift.

[51] See Walsh (1901; 389), (1921; 540).

An advantage of the Mixed TPD and Jevons index is that it can be implemented in real time without revision or linking problems. However, a disadvantage of $P_{TPDJ}{}^{t}$ is that the seasonal pattern of prices that occurred in the first year of "training" data will persist in subsequent periods. If there are changing seasonal patterns, then this property of the method may be problematic. It could be addressed by periodically changing the base year of training data and then starting a new set of indexes. Furthermore, the seasonal patterns to prices could be subject to more or less random fluctuations. In order to address this randomness problem, the time period dummy method could be implemented using two or more years of training data rather than just using a single year's data. This could lead to a more representative set of seasonal factors. We implemented this modification using our empirical data set.

The final *Blended TPD and Jevons index*, $P_{TPDJ}{}^{t*}$, is defined as follows. (i) Compute the Time Product Dummy indexes for a historical data set that consists of 24 consecutive months. Call the resulting indexes $P_{TPD}{}^{t*}$ for m = 1,...,24. (ii) Set $P_{TPDJ}{}^{t*}$ for the first 12 months equal to the corresponding Time Product Dummy indexes so that $P_{TPDJ}{}^{t*} = P_{TPD}{}^{t*}$ for t = 1,...,12. (ii) For subsequent months, use the year over year fixed base maximum overlap Jevons indexes $P_{JFm}{}^{y*}$ to link month m in year y ≥ 2 to $P_{TPDJ}{}^{m}$. Thus repeating our earlier description, for our empirical example, for year y = 2, we have $P_{TPDJ}{}^{12+m*} = P_{TPDJ}{}^{m*}\times P_{JFm}{}^{2*}$ for m = 1,...,12; for year y = 3, we have $P_{TPDJ}{}^{24+m*} = P_{TPDJ}{}^{m*}\times P_{JFm}{}^{3*}$ for m = 1,...,12; ... and for year y = 6, we have $P_{TPDJ}{}^{60+m*} = P_{TPDJ}{}^{m*}\times P_{JFm}{}^{6*}$ for m = 1,...,12. The blended indexes $P_{TPDJ}{}^{t*}$ are listed in Table 22 and plotted on Chart 11.

The final elementary index that we consider in this section is an adaptation of the Predicted Share multilateral index $P_S{}^{t*}$ that was defined in the previous section. Since in the present section, we are considering price indexes that depend solely on price information, in place of a maximum overlap bilateral Fisher index to link the prices of two months, the maximum overlap bilateral Jevons index $P_J{}^{*}(t/r)$ defined by (113) above will be used to relate the prices of the current month to a previous month that has the lowest measure of relative price dissimilarity. In the previous section, the Predicted Share measure of relative price dissimilarity between the prices of two months was defined by (105). This definition depended on the availability of quantity (or expenditure) information but in the present context, only price information is available. When quantity and expenditure information is not available, it is natural to assume that either quantities purchased in a month or expenditures on available products are equal. The assumption of equal quantities depends on units of product measurement, which are to some extent arbitrary and so we will make the assumption of equal expenditures on available products in each month. This assumption is equivalent to an assumption that expenditure shares on available commodities in a month are equal.

Recall that the price of product n in month t is denoted by $p_{t,n}$ for n = 1,...,N and t = 1,...,T where T = YM, Y is the number of years in the sample and M is the number of months in a year. If product n in month t was not available (i.e., not purchased by the households in scope), then $p_{t,n}$ is set equal to 0. The vector of month t prices is $p^t \equiv [p_{t,1},...,p_{t,N}]$ for t = 1,...,T. The set of available products in month t is S(t) and the number of available products in month t is N(t) for t = 1,...,T. The set of products that are available in both months t and r is the intersection of the sets S(t) and S(r), denoted by $S(t)\cap S(r)$. The number of matched products that are available in both months t and r is N(t,r). If there are no unmatched products in months t and r, then N(t) = N(r) = N(t,r). We assume that there is at least one matched product between every pair of months in the sample.

The *imputed quantities*, $q_{t,n}$, that will generate equal expenditure shares for products n that are present in month t are defined as follows for t = 1,...,T:

$$(121)\ q_{t,n} \equiv 1/p_{t,n}N(t) \text{ if } n\in S(t); \qquad \equiv 0 \text{ if } n\notin S(t).$$

The imputed expenditure share for product n in month t is $s_{t,n} \equiv p_{t,n}q_{t,n}/p^t\cdot q^t$ for t = 1,...,T and n = 1,...,N. Using the $q_{t,n}$ defined by (121), these expenditure shares are equal to the following expressions for t = 1,...,T:

$$(122)\ s_{t,n} = 1/N(t) \text{ if } n\in S(t); \qquad = 0 \text{ if } n\notin S(t).$$

To form month t predicted shares, use the prices of month r and the (imputed) quantities of month t to form the following *predicted shares*, $s_{t,r,n}$:

$$(123)\ s_{t,r,n} \equiv p_{r,n}q_{t,n}/p^r\cdot q^t\ ; \qquad t = 1,\ldots,T;\ r = 1,\ldots,T;\ n = 1,\ldots,N.$$

If product n is not available in month t, so that $n\notin S(t)$, then $q_{t,n} = 0$ and the following equations hold:

$$(124)\ s_{t,n} = s_{t,r,n} = 0\ ; \qquad t = 1,\ldots,T;\ r = 1,\ldots,T;\ n\notin S(t).$$

If product n is available in month t, so that $n\in S(t)$, then $q_{t,n} > 0$, $p_{t,n} > 0$, $s_{t,n} = 1/N(t)$ and the predicted shares $s_{t,r,n}$ satisfy the following equations:

$$\begin{aligned}(125)\ s_{t,r,n} &\equiv p_{r,n}q_{t,n}/p^r\cdot q^t\ ; && t = 1,\ldots,T;\ r = 1,\ldots,T;\ n\in S(t)\\ &= [p_{r,n}/p_{t,n}N(t)]/\Sigma_{k\in S(t)}\ [p_{r,k}/p_{t,k}N(t)] && \text{using (121)}\\ &= [p_{r,n}/p_{t,n}]/\Sigma_{k\in S(t)}\ [p_{r,k}/p_{t,k}]\\ &= [p_{r,n}/p_{t,n}]/\Sigma_{k\in S(t)\cap S(r)}\ [p_{r,k}/p_{t,k}]\end{aligned}$$

where the last equality follows from the fact that $p_{r,k} = 0$ if k does not belong to S(r); i.e., if $k\notin S(r)$.

The *predicted share error* $e_{t,r,n}$ in using $s_{t,r,n}$ to predict $s_{t,n}$ is defined as follows:

$$(126)\ e_{t,r,n} \equiv s_{t,n} - s_{t,r,n}\ ; \qquad t = 1,\ldots,T;\ r = 1,\ldots,T;\ n = 1,\ldots,N.$$

Using definitions (122) and (123), it is straightforward to show that the sum over products n of the prediction errors $e_{t,r,n}$ is equal to 0 for each pair of months, r and t; i.e., we have:

$$(127)\ \Sigma_{n=1}^{N}\ e_{t,r,n} = 0\ ; \qquad t = 1,\ldots,T;\ r = 1,\ldots,T.$$

Note that if product n is not available in month t, then using (122) and (124), it can be seen that the predicted error $e_{t,r,n}$ will equal 0; i.e., we have the following equalities:

$$(128)\ e_{t,r,n} = 0\ ; \qquad t = 1,\ldots,T;\ r = 1,\ldots,T;\ n\notin S(t).$$

Using (127) and (128), it can be seen that the mean of the predicted errors $e_{t,r,n}$ over all products that are available in month t is equal to zero; i.e., we have:

$$(129)\ [1/N(t)]\Sigma_{n\in S(t)}\, e_{t,r,n} = 0\ ; \qquad t = 1,...,T;\ r = 1,...,T.$$

Using only price information, the *Predicted Share measure of relative price dissimilarity* between months t and r is $\Delta_{PS}(p^t,p^r)$ defined as follows:

$$(130)\ \Delta_{PS}(p^t,p^r) \equiv \sigma_{t,r} + \sigma_{r,t}\ ; \qquad t = 1,...,T;\ r = 1,...,T$$

where $\sigma_{t,r}$ is defined as[52]

$$\begin{aligned}(131)\ \sigma_{t,r} &\equiv \Sigma_{n=1}^{N}\, e_{t,r,n}^{2}\ ; && t = 1,...,T;\ r = 1,...,T\\ &= \Sigma_{n\in S(t)}\, e_{t,r,n}^{2} && \text{using (128)}\\ &= \Sigma_{n\in S(t)}\, \{s_{t,n} - s_{t,r,n}\}^2 && \text{using (126)}\\ &= \Sigma_{n\in S(t),n\notin S(r)}\, \{[1/N(t)] - 0\}^2 + \Sigma_{n\in S(t)\cap S(r)}\, \{[1/N(t)] - x_{t,r,n}\}^2 && \text{using (122) and (125)}\\ &= \{[N(t) - N(t,r)]/N(t)^2\} + \Sigma_{n\in S(t)\cap S(r)}\, \{[1/N(t)] - x_{t,r,n}\}^2. &&\end{aligned}$$

The $x_{t,r,n}$ are *normalized price ratios for matched products* present in periods t and r and are defined as follows :

$$(132)\ x_{t,r,n} \equiv (p_{r,n}/p_{t,n})/\Sigma_{k\in S(t)\cap S(r)}\, (p_{r,k}/p_{t,k})\ ; \qquad t = 1,...,T;\ r = 1,...,T;\ n\in S(t)\cap S(r).$$

If prices in months t and r are equal (or proportional so that $p^t = \lambda p^r$ for some scalar $\lambda > 0$), then $N(t) = N(r) = N(t,r) = N(r,t)$ and $x_{t,r,n} = 1/N(t,r) = 1/N(t)$. In this case, $\sigma_{t,r} = \sigma_{r,t} = 0$ and thus $\Delta_{PS}(p^t,p^r) = 0$. In general, $\Delta_{PS}(p^t,p^r) \geq 0$, $\Delta_{PS}(p^t,p^r) = \Delta_{PS}(p^r,p^t)$ (symmetry property) and $\Delta_{PS}(p^t,p^r) \leq 2$.

Note that $[N(t) - N(t,r)]/N(t)^2 \geq 0$ is a *penalty term* that is positive if available products purchased in months t and r are not matched. If the list of available products in months t and r is identical, then $N(t) = N(r) = N(t,r) = N(r,t)$ and this penalty term is equal to 0. If products are matched in months t and r, then $S(t) = S(r) = S(t)\cap S(r)$ and the second set of terms in the last equality of (131) becomes $\Sigma_{n\in S(t)}\, \{[1/N(t)] - x_{t,r,n}\}^2$ which, using (129), is proportional to the *variance* of the normalized price ratios, $x_{t,r,n}$. Below, we will obtain a decomposition of this second set of terms in the case where products are not matched between months t and r.

In the general case where prices in month t are not necessarily proportional to prices in month r, we calculate the mean and variance of the $x_{t,r,n}$ over products n that are present in months t and r. It turns out that the sum of the $x_{t,r,n}$ over products n that are present in both months t and r is always equal to 1; i.e., we have the following equalities using definitions (132):

$$\begin{aligned}(133)\ \Sigma_{n\in S(t)\cap S(r)}\, x_{t,r,n} &= \Sigma_{n\in S(t),n\notin S(r)}\, (p_{r,n}/p_{t,n})/\Sigma_{k\in S(t)\cap S(r)}\, (p_{r,k}/p_{t,k})\ ; && t = 1,...,T;\ r = 1,...,T\\ &= 1. &&\end{aligned}$$

The number of common products that are present in months t and r is N(t,r). Thus the *mean* of the $x_{t,r,n}$ over the common products n that are present in months t and r is $\mu_{t,r}$ defined as follows:

$$\begin{aligned}(134)\ \mu_{t,r} &\equiv \Sigma_{n\in S(t)\cap S(r)}\, x_{t,r,n}/N(t,r)\ ; && t = 1,...,T;\ r = 1,...,T\\ &= 1/N(t,r) && \text{using (133).}\end{aligned}$$

---

[52] The $\sigma_{r,t}$ are defined by (131) and (132) by simply interchanging t and r.

Note that (134) implies that :

(135) $\Sigma_{n\in S(t)\cap S(r)} [x_{t,r,n} - \mu_{t,r}] = 0$; $\qquad t = 1,...,T; r = 1,...,T.$

A *measure of relative price dissimilarity* between the common product prices of months t and r is $\delta_{t,r}$ defined as follows:

(136) $\delta_{t,r} \equiv \Sigma_{n\in S(t)\cap S(r)} [x_{t,r,n} - \mu_{t,r}]^2$; $\qquad t = 1,...,T; r = 1,...,T.$

It can be seen that $\delta_{t,r}$ is proportional to the *variance* of the normalized price ratios $x_{t,r,n}$ over products that are present in both months t and r. If the prices of products that are present in both months t and r are identical or proportional to each other, it can be verified that $\delta_{t,r}$ is equal to 0.

Using equations (131), we have the following decomposition for $\sigma_{t,r}$:

$$
\begin{aligned}
(137)\ \sigma_{t,r} &= \{[N(t) - N(t,r)]/N(t)^2\} + \Sigma_{n\in S(t)\cap S(r)} \{[1/N(t)] - x_{t,r,n}\}^2 ; \qquad t = 1,...,T; r = 1,...,T \\
&= \{[N(t) - N(t,r)]/N(t)^2\} + \Sigma_{n\in S(t)\cap S(r)} \{[1/N(t)] - \mu_{t,r} - [x_{t,r,n} - \mu_{t,r}]\}^2 \\
&= \{[N(t) - N(t,r)]/N(t)^2\} + \Sigma_{n\in S(t)\cap S(r)} \{[1/N(t)] - \mu_{t,r}\}^2 + \Sigma_{n\in S(t)\cap S(r)} \{x_{t,r,n} - \mu_{t,r}\}^2 \\
&\qquad \text{using (135)} \\
&= \{[N(t) - N(t,r)]/N(t)^2\} + \Sigma_{n\in S(t)\cap S(r)} \{[1/N(t)] - [1/N(t,r)]\}^2 + \delta_{t,r} \quad \text{using (134) and (136)} \\
&= \{[N(t) - N(t,r)]/N(t)^2\} + N(t,r)\{[1/N(t)] - [1/N(t,r)]\}^2 + \delta_{t,r} \\
&= [1/N(t,r)] - [1/N(t)] + \delta_{t,r}.
\end{aligned}
$$

The following decomposition for $\sigma_{r,t}$ can be derived in an analogous fashion:

(138) $\sigma_{r,t} = [1/N(r,t)] - [1/N(r)] + \delta_{r,t}$; $\qquad t = 1,...,T; r = 1,...,T.$

Using (137), (138) and definitions (130), the *Predicted Share measure of relative price dissimilarity* between months t and r has the following decomposition:

$$
\begin{aligned}
(139)\ \Delta_{PS}(p^t,p^r) &\equiv \sigma_{t,r} + \sigma_{r,t} ; \qquad t = 1,...,T; r = 1,...,T \\
&= [1/N(t,r)] - [1/N(t)] + \delta_{t,r} + [1/N(r,t)] - [1/N(r)] + \delta_{r,t} \\
&= [2/N(t,r)] - [1/N(t)] - [1/N(r)] + \delta_{t,r} + \delta_{r,t}
\end{aligned}
$$

where the last equality follows from the fact that $N(t,r) = N(r,t)$. It can be seen that the term $[2/N(t,r)] - [1/N(t)] - [1/N(r)] \geq 0$ is the *total penalty for a possible lack of matching of the overlap prices* in months t and r. If the products that are present in month t are also present in month r, then $N(t) = N(r) = N(t,r)$ and there is no penalty for a lack of matching. The term $\delta_{t,r}$ is proportional to the variance of the normalized matched relative prices $p_{r,n}/p_{t,n}$ and the term $\delta_{r,t}$ is proportional to the variance of the normalized matched reciprocal relative prices $p_{t,n}/p_{r,n}$. If prices in months r and t are equal or proportional, then the penalty for a lack of matching is 0 and the two matched relative price dissimilarity terms $\delta_{t,r}$ and $\delta_{r,t}$ are also equal to 0. It should be noted that the *actual unmatched prices* in periods t and s *do not play a role* in the measure of relative price dissimilarity between the prices of months t and r. However, the *number of unmatched prices* does play a role.

The problem of trading off a lack of matching of prices between two periods and the dispersion in the matched prices is a difficult one.[53] The modified predicted share methodology explained above does accomplish this tradeoff but further research may find more direct methods for making this tradeoff. What is clear is that a method for linking bilateral indexes needs to take into account the lack of matching of prices and the method should be based on some principles. The principle used in the above method was to use the prices of month r and the quantities of month t to predict the imputed expenditure shares of period t. In the future, "better" principles may be found.

The entire set of Modified Predicted Share Dissimilarity measures for our empirical example is a 72 by 72 element (symmetric) matrix. Table 22 below lists the first 12 rows and columns of the matrix of the bilateral measures of Modified Predicted Share Price Dissimilarity for our empirical example.

Table 22: Month to Month Modified Predicted Share Measures of Price Dissimilarity Using Zeros for Missing Prices

r,t	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0.0189	0.0195	0.0499	0.1517	0.1644	0.4155	0.4118	0.2229	0.0458	0.0366	0.0037
2	0.0189	0	0.0009	0.0186	0.1004	0.1852	0.4470	0.4488	0.2515	0.0676	0.0597	0.0263
3	0.0195	0.0009	0	0.0182	0.1026	0.1875	0.4532	0.4565	0.2537	0.0686	0.0610	0.0275
4	0.0499	0.0186	0.0182	0	0.0828	0.1666	0.4291	0.4312	0.2395	0.1019	0.0950	0.0612
5	0.1517	0.1004	0.1026	0.0828	0	0.0439	0.1845	0.2001	0.3021	0.1368	0.2001	0.1656
6	0.1644	0.1852	0.1875	0.1666	0.0439	0	0.0673	0.1246	0.1956	0.0989	0.1358	0.1716
7	0.4155	0.4470	0.4532	0.4291	0.1845	0.0673	0	0.0276	0.0739	0.1122	0.1723	0.4144
8	0.4118	0.4488	0.4565	0.4312	0.2001	0.1246	0.0276	0	0.0582	0.1033	0.1836	0.4012
9	0.2229	0.2515	0.2537	0.2395	0.3021	0.1956	0.0739	0.0582	0	0.0447	0.0871	0.2173
10	0.0458	0.0676	0.0686	0.1019	0.1368	0.0989	0.1122	0.1033	0.0447	0	0.0149	0.0451
11	0.0366	0.0597	0.0610	0.0950	0.2001	0.1358	0.1723	0.1836	0.0871	0.0149	0	0.0320
12	0.0037	0.0263	0.0275	0.0612	0.1656	0.1716	0.4144	0.4012	0.2173	0.0451	0.0320	0

The set of real time links which minimize the above dissimilarity measures for the first 12 observations are as follows:[54]

1 – 2 – 3 – 4 – 5 – 6 – 7 – 8 – 9 – 10 – 11
|
12.

The maximum overlap bilateral Jevons indexes $P_J^*(t/r)$ defined by (113) above are used to link the prices of month t to a prior month r. It can be seen that the new set of bilateral links is the set of links that generates the chained maximum overlap Jevons indexes for months 1 to 11. Thus the *Similarity Linked Maximum Overlap Jevons index*, P_{SJ}^{t*}, equals P_{JCH}^{t*}, the Maximum Overlap Chained Jevons index defined by (116) and (117) above, for t = 1,...,11. However, month 12 is linked directly to month 1. Thus $P_{SJ}^{12*} = P_{SJ}^{1*} \times P_J^*(12/1)$. The remainder of the Similarity Linked Maximum Overlap Jevons indexes are constructed in the same manner that was used to construct the Predicted Share Similarity Linked indexes P_S^{t*} except that the new matrix of dissimilarity measures $\Delta_{PS}(p^t,p^r)$ is used in place of the previous matrix of Predicted Share Dissimilarity measures $\Delta_{PS}(p^r,p^t,q^r,q^t)$ defined by (106) and the maximum overlap bilateral Fisher indexes that were used to link the prices of months in real time to the prices of previous months are replaced

[53] For additional discussions of this issue see Hill and Timmer (2006). For additional measures of relative price dissimilarity for matched prices, see Diewert (2009)
[54] This set of bilateral links is almost the same as the set of links that were used to link the first 12 observations of the Predicted Share indexes P_S^{t*}; see the links listed below Table 16 in section 7 above.

by maximum overlap bilateral Jevons indexes. Again, it turns out that the set of bilateral links for the first 12 months basically determines the seasonal fluctuations for the similarity linked indexes P_S^{t*} for the remainder of the sample.[55] The Similarity Linked Maximum Overlap Jevons indexes, P_{SJ}^{t*}, are listed in Table 23 below.

Table 23 : Similarity Linked Indexes, Maximum Overlap Jevons Fixed Base and Chained Indexes and Time Product Dummy Indexes

t	P_S^{t*}	P_{SJ}^{t*}	P_{JFB}^{t*}	P_{JCH}^{t*}	P_{TPD}^{t}	P_{TPDJ}^{t*}
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.06603	1.03812	1.03812	1.03812	1.02229	1.02743
3	1.17647	1.10519	1.08161	1.10519	1.08834	1.09381
4	1.17956	1.12441	1.09864	1.12441	1.10026	1.10881
5	1.18310	1.15614	1.13498	1.15614	1.24733	1.23499
6	1.00296	1.05968	1.13521	1.05968	1.14674	1.13568
7	1.01198	1.10518	1.31939	1.10518	1.23890	1.20161
8	1.05554	1.22312	1.78827	1.22312	1.43385	1.38662
9	0.97973	1.11168	1.40705	1.11168	1.28879	1.26808
10	0.99067	1.07692	1.27978	1.07692	1.25377	1.24421
11	1.04107	0.95033	1.07188	0.95033	1.10326	1.10361
12	0.97592	1.00749	1.00749	0.89324	1.00749	1.00749
13	0.99684	1.02568	1.02568	0.90936	1.02568	1.02568
14	1.17902	1.11710	1.06090	0.94059	1.10007	1.10560
15	1.08056	1.07900	1.08262	0.90851	1.06255	1.06789
16	1.17474	1.16270	1.16025	0.99405	1.13772	1.14657
17	1.10498	1.16209	1.27798	1.08509	1.32010	1.24134
18	1.30841	1.40187	1.40174	1.21125	1.52851	1.50241
19	1.18142	1.45397	1.48595	1.33856	1.59591	1.58084
20	1.23391	1.47303	1.47735	1.35611	1.50812	1.54295
21	1.09986	1.17793	1.32928	1.27802	1.36559	1.34364
22	1.23179	1.15592	1.26980	1.27785	1.34194	1.33548
23	1.06906	1.13275	1.13275	1.13993	1.13275	1.16629
24	1.04392	1.10289	1.10289	1.10988	1.10289	1.10289
25	1.02270	1.08747	1.08747	1.09436	1.08747	1.08747
26	1.22856	1.16556	1.10175	1.10873	1.14779	1.15355
27	1.17215	1.15798	1.14475	1.10153	1.14033	1.14606
28	1.25327	1.22744	1.22192	1.18373	1.20107	1.21042
29	1.22223	1.40403	1.35287	1.33273	1.48657	1.34725
30	1.15449	1.34776	1.48215	1.27956	1.45849	1.37485
31	1.20526	1.34029	1.64760	1.37706	1.50247	1.46191
32	1.16278	1.34423	1.82280	1.38112	1.50689	1.55295
33	1.18929	1.23171	1.37302	1.23008	1.42795	1.40500
34	1.31066	1.18542	1.33810	1.21013	1.37619	1.36957
35	1.07810	1.03496	1.12963	1.07487	1.17947	1.16308
36	1.01195	1.06487	1.06487	1.01325	1.06487	1.06487
37	1.01076	1.04600	1.04600	0.99529	1.04600	1.04600

[55] The remainder of the real time maximum overlap predicted share bilateral Jevons index links for the next 60 months are as follows: 13/12, 14/3, 15/2, 16/4, 17/5, 18/6, 19/7, 20/19, 21/9, 22/11, 23/12 and 24/23, 25/24, 26/14, 27/2, 28/4, 29/6, 30/6, 31/7, 32/7, 33/21, 34/22, 35/22, 36/24, 37/36, 38/26, 39/27, 40/28, 41/29, 42/30, 43/31, 44/20, 45/21, 46/10, 47/22, 48/25, 49/48, 50/26, 51/40, 52/51, 53/29, 54/42, 55/43, 56/20, 57/21, 58/35, 59/25, 60/59, 61/59, 62/38, 63/39, 64/40, 65/41, 66/54, 67/43, 68/44, 69/9, 70/46, 71/22 and 72/48. Most of these bilateral links link the same months as were used to construct P_S^{t*}.

38	**1.16812**	**1.12268**	**1.06566**	**1.01400**	**1.10557**	**1.11112**
39	**1.17108**	**1.14508**	**1.12395**	**1.03423**	**1.12762**	**1.13329**
40	**1.39663**	**1.33105**	**1.33629**	**1.20021**	**1.30246**	**1.31258**
41	**1.50841**	**1.66272**	**1.56307**	**1.37630**	**1.76046**	**1.59547**
42	**1.37756**	**1.52866**	**1.92004**	**1.25089**	**1.68208**	**1.55939**
43	**1.22151**	**1.41849**	**2.15585**	**1.15706**	**1.56196**	**1.54720**
44	**1.05506**	**1.43013**	**1.89554**	**1.06246**	**1.46420**	**1.49802**
45	**1.22173**	**1.30768**	**1.52287**	**1.08274**	**1.51602**	**1.49165**
46	**1.19999**	**1.28729**	**1.47388**	**1.06558**	**1.49868**	**1.47337**
47	**1.26828**	**1.16789**	**1.29467**	**0.95233**	**1.31599**	**1.33300**
48	**1.13921**	**1.19473**	**1.19473**	**0.87882**	**1.19473**	**1.19473**
49	**1.13475**	**1.17935**	**1.17935**	**0.86751**	**1.17935**	**1.17935**
50	**1.38339**	**1.29628**	**1.23215**	**0.90634**	**1.27652**	**1.28293**
51	**1.29063**	**1.24180**	**1.23563**	**0.87403**	**1.21512**	**1.22698**
52	**1.43303**	**1.35551**	**1.35948**	**0.95406**	**1.32639**	**1.33670**
53	**1.34386**	**1.53671**	**1.52251**	**1.03014**	**1.62705**	**1.47457**
54	**1.25757**	**1.49143**	**1.92473**	**1.00391**	**1.64111**	**1.52141**
55	**1.34547**	**1.52037**	**2.05735**	**1.11495**	**1.67415**	**1.65834**
56	**1.30412**	**1.61240**	**1.71865**	**1.10020**	**1.65081**	**1.68895**
57	**1.26875**	**1.33240**	**1.46566**	**1.08668**	**1.54468**	**1.51985**
58	**1.34737**	**1.29692**	**1.45182**	**1.05901**	**1.47800**	**1.46867**
59	**1.09738**	**1.21017**	**1.21017**	**0.88275**	**1.21017**	**1.24599**
60	**1.01922**	**1.11805**	**1.11805**	**0.81555**	**1.11805**	**1.11804**
61	**1.07767**	**1.14744**	**1.14744**	**0.83699**	**1.14744**	**1.14744**
62	**1.39115**	**1.29232**	**1.21897**	**0.88916**	**1.27262**	**1.27901**
63	**1.32072**	**1.29483**	**1.29046**	**0.89089**	**1.27510**	**1.26479**
64	**1.39001**	**1.34113**	**1.36739**	**0.93338**	**1.31231**	**1.32252**
65	**1.52597**	**1.73864**	**1.62717**	**1.05969**	**1.73957**	**1.59232**
66	**1.25740**	**1.48917**	**1.91313**	**0.97614**	**1.63863**	**1.51910**
67	**1.22459**	**1.46295**	**2.18107**	**1.01502**	**1.61092**	**1.59570**
68	**1.11160**	**1.52475**	**1.98518**	**0.98081**	**1.56108**	**1.59714**
69	**1.27951**	**1.42959**	**1.74583**	**0.97806**	**1.65734**	**1.63071**
70	**1.27885**	**1.36176**	**1.54445**	**0.94301**	**1.58539**	**1.54454**
71	**1.23088**	**1.17084**	**1.29409**	**0.81873**	**1.35927**	**1.33240**
72	**1.19115**	**1.25093**	**1.25093**	**0.79142**	**1.25093**	**1.25093**
Mean	**1.18920**	**1.25460**	**1.36900**	**1.06470**	**1.32860**	**1.31120**

From Table 23, it can be seen that the chained Jevons index, $P_{JCH}{}^{t*}$, that uses maximum overlap bilateral Jevons indexes suffers from a considerable amount of downward chain drift so it cannot be recommended for use. The "best" index that uses price and quantity information, the Predicted Share Similarity linked index that uses maximum overlap bilateral Fisher indexes as basic building blocks, $P_S{}^{t*}$, finished about 6 percentage points below the remaining four indexes, $P_{SJ}{}^{t*}$, $P_{JFB}{}^{t*}$, $P_{TPD}{}^{t*}$ and $P_{TPDJ}{}^{t*}$. These four indexes cannot control adequately for substitution bias (since they depend only on price information) whereas $P_S{}^{t*}$ does deal adequately with substitution bias. Thus if a statistical agency is forced to rely on price data alone for an elementary index that has strongly seasonal commodities, then there is a strong likelihood that some substitution bias will occur which can be substantial.

The four indexes, $P_{SJ}{}^{t*}$, $P_{JFB}{}^{t*}$, $P_{TPD}{}^{t*}$ and $P_{TPDJ}{}^{t*}$, all finished at exactly the same value in month 72. However, their means were substantially different in some cases. The standard Time Product Dummy indexes, $P_{TPD}{}^{t*}$, and the Modified Time Product Dummy indexes that used year over year maximum overlap Jevons indexes as well, $P_{TPDJ}{}^{t*}$, had means of 1.3286 and 1.3112 respectively. As can be seen in Chart 11, these two indexes approximated each other rather well. The mean of the Jevons maximum overlap fixed base indexes, P_{JFB}^{t*}, was the highest of the six indexes at 1.3690. Chart 11 shows that this high mean is due to very large upward seasonal fluctuations for months in the middle of the year, where product matches with the products available in January were very low. The price index (over the five indexes that used only price information) that best approximated the Predicted Share index P_S^{t*} is the Similarity Linked Maximum Overlap Jevons index, P_{SJ}^{t*}.

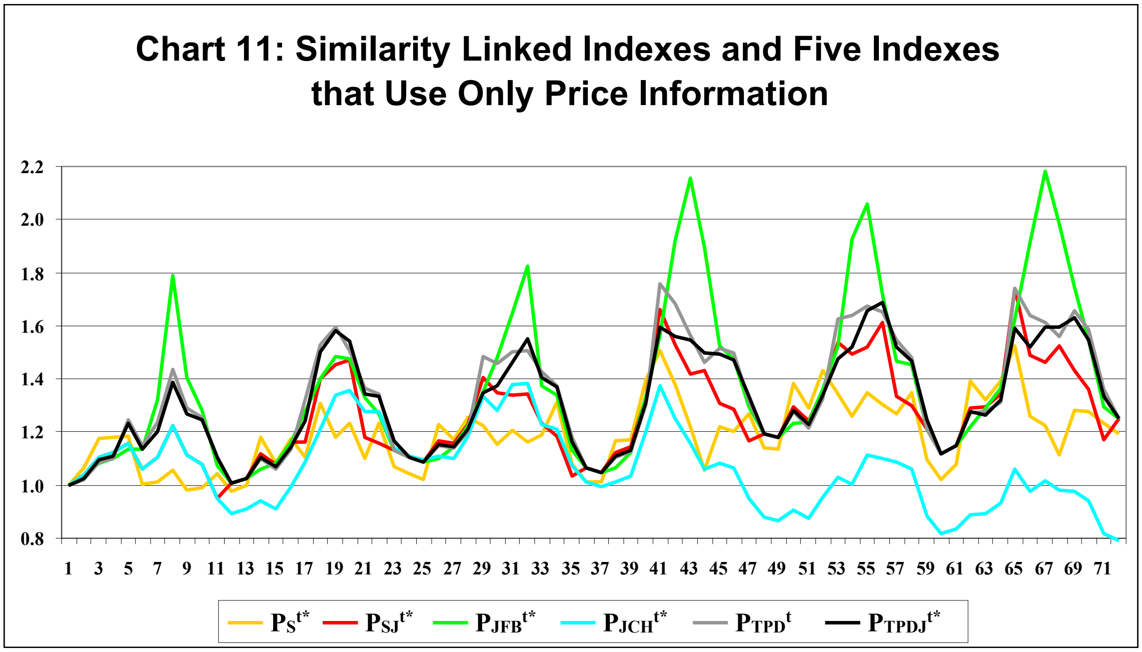


Chart 11 shows that P_S^{t*} has by far the smallest seasonal variations. Relative to this preferred index, the chained Jevons index, P_{JCH}^{t*}, has a large downward bias and the fixed base Jevons index, P_{JFB}^{t*}, has a large upward bias on average due to its huge seasonal fluctuations. The remaining three indexes, P_{SJ}^{t*}, P_{TPD}^{t} and P_{TPDJ}^{t*}, finish at the same point which is 6 percentage points above P_S^{72*}. These three indexes are fairly close to each other but the similarity linked Jevons index, P_{SJ}^{t*}, tends to lie below the two Time Product Dummy indexes, P_{TPD}^{t} and P_{TPDJ}^{t*}, and P_{SJ}^{t*} has smaller seasonal fluctuations. Overall, for our particular example, P_{SJ}^{t*} provides the best approximation to our preferred index, P_S^{t*}.[56]

In the following section, indexes which use annual baskets or annual expenditure shares will be studied.

10. Annual Basket Lowe Indexes and Annual Share Weighted Young Indexes

For many consumer expenditure categories, national statistical agencies are not able to collect price and quantity information for many if not most expenditure categories. Instead, they collect a sample of prices in real time and collect annual household expenditures by broad category on a

[56] The correlation coefficients between P_S^{t*} and P_{SJ}^{t*}, P_{JFB}^{t*}, P_{JCH}^{t*}, P_{TPD}^{t*}, P_{TPDJ}^{t*} are 0.730, 0.392, 0.137, 0.620 and 0.602 respectively. The correlation coefficients between P_{SJ}^{t*} and P_{TPD}^{t*}, P_{TPDJ}^{t*} are 0.913, 0.898 respectively. Thus these three indexes approximate each other reasonably well.

delayed basis using a consumer expenditure survey. Thus for many expenditure categories, statistical agencies construct either a Lowe (1823) or Young (1812) index. These indexes use a combination of annual expenditures on a past year and current month information on prices. In this section, we will construct versions of these indexes using our seasonal products data for Israel.

When constructing a price index for a category of household goods and services using annual expenditure data from a past period, statistical agencies have to deal with missing prices for strongly seasonal commodities (commodities that are not available for all seasons of the year). Agencies solve this problem by using carry forward prices for the missing products.[57] Thus in this section, for our empirical example, we again use the monthly price data that are listed in Table A23 in the Appendix. This table has the carry forward/backward prices for products that are missing in any month.

The month t price for product n is defined as $p_{t,n}$ for t = 1,...,T and n = 1,...,N where T = MY and M is the number of months in the year and Y is the number of years in the sample of prices. The notation for quantities is the same as was used in section 2 above: $q_{y,m,n}$ is the quantity of product n that is purchased by households in month m of year y where y = 1,...,Y; m = 1,...,M and n = 1,...,N. *Annual quantities of product n* purchased in year y, $q_{A,y,n}$, are obtained by summing purchases of product n in year y over the months in the year; i.e., we have the following definitions:

$$(140)\quad q_{A,y,n} \equiv \Sigma_{m=1}^{M} q_{y,m,n}\,; \qquad y = 1,\ldots,Y;\ n = 1,\ldots,N.$$

Using the annual basket weights for year 1 in our sample (the $q_{a,1,n}$ defined by (140)), and the prices of month t, $p_{t,n}$, the *Fixed Base Lowe index* for month t using the weights of year 1, $P_{LO1}{}^{t}$, is defined as follows:[58]

$$(141)\quad P_{LO1}{}^{t} \equiv \Sigma_{n=1}^{N} p_{t,n}q_{A,1,n}/\Sigma_{n=1}^{N} p_{1,n}q_{A,1,n}\,; \qquad t = 1,\ldots,T.$$

Using the data listed in Tables A23 and A24, these fixed base Lowe indexes are listed in Table 24 below and plotted in Chart 12.[59]

Some statistical agencies use the annual weights of the prior year and use a new Lowe index for 12 months to update the prior year's indexes. To approximate this type of index, we construct a *Lagged One Year Chained Lowe index* for month t, $P_{LO2}{}^{t}$, as follows. For t = 1,...,24, define $P_{LO2}{}^{t} \equiv P_{LO1}{}^{t}$. Thus we use the annual quantities for year 1 for the first 24 months of data. For a month t = 25,...,36 in the third year, define a new *link Lowe index* using the weights of year 2 and the prices of year 3 relative to December of year 2, $[\Sigma_{n=1}^{N} p_{t,n}q_{A,2,n}/\Sigma_{n=1}^{N} p_{24,n}q_{A,2,n}]$, and then link this index to the index value for $P_{LO2}{}^{t}$ at t = 24. Thus for t = 25,...,36, define $P_{LO2}{}^{t} \equiv P_{LO2}{}^{24}\times[\Sigma_{n=1}^{N} p_{t,n}q_{A,2,n}/\Sigma_{n=1}^{N} p_{24,n}q_{A,2,n}]$. For year 4, use the quantity weights of year 3 and the new Lowe link index that compares the prices of month t in year 4 to month 12 in year 3 to extend the definition of $P_{LO2}{}^{t}$. Thus for t = 37,...,48, define $P_{LO2}{}^{t} \equiv P_{LO2}{}^{36}\times[\Sigma_{n=1}^{N} p_{t,n}q_{A,3,n}/\Sigma_{n=1}^{N} p_{36,n}q_{A,3,n}]$. In a similar manner, for t = 49,...,60, define $P_{LO2}{}^{t} \equiv P_{LO2}{}^{48}\times[\Sigma_{n=1}^{N} p_{t,n}q_{A,4,n}/\Sigma_{n=1}^{N} p_{48,n}q_{A,4,n}]$ and for t = 61,...,72,

[57] Other methods for imputing the missing prices are also used.

[58] In the context of seasonal price indexes, this type of index corresponds to Bean and Stine's (1924; 31) Type A index.

[59] The year 1 annual quantity weights $q_{A,1,n}$ are as follows for our sample of 14 types of fruit: 7.968, 7.159, 27.106, 2.285, 0.966, 8.805, 10.069, 2.266, 0.664, 0.884, 3.560, 9.528, 0.782, 2.168.

define $P_{LO2}{}^t \equiv P_{LO2}{}^{60}\times[\Sigma_{n=1}{}^N p_{t,n}q_{A,5,n}/\Sigma_{n=1}{}^N p_{60,n}q_{A,5,n}]$. These chained Lowe indexes using annual weights lagged one year, $P_{LO2}{}^t$, are listed in Table 24 below and plotted in Chart 12.

Some countries use annual weights that are lagged two years. To approximate this type of index, we construct a *Lagged Two Year Chained Lowe index* for month t, $P_{LO3}{}^t$, as follows. For t = 1,...,36, define $P_{LO3}{}^t \equiv P_{LO1}{}^t$. Thus we use the annual quantities for year 1 for the first 36 months of data to construct this alternative Lowe index which will be equal to the fixed base Lowe index, $P_{LO1}{}^t$, for the first three years of data. For the fourth year of data, define a new *link Lowe index* using the weights of year 2 and the prices of year 4 relative to December of year 3, $[\Sigma_{n=1}{}^N p_{t,n}q_{A,2,n}/\Sigma_{n=1}{}^N p_{36,n}q_{A,2,n}]$, and then link this index to the index value for $P_{LO3}{}^t$ at t = 36. Thus for t = 37,...,48, define $P_{LO3}{}^t \equiv P_{LO3}{}^{36}\times[\Sigma_{n=1}{}^N p_{t,n}q_{A,2,n}/\Sigma_{n=1}{}^N p_{36,n}q_{A,2,n}]$. In a similar manner, for t = 49,...,60, define $P_{LO3}{}^t \equiv P_{LO3}{}^{48}\times[\Sigma_{n=1}{}^N p_{t,n}q_{A,3,n}/\Sigma_{n=1}{}^N p_{48,n}q_{A,3,n}]$ and for t = 61,...,72, define $P_{LO3}{}^t \equiv P_{LO3}{}^{60}\times[\Sigma_{n=1}{}^N p_{t,n}q_{A,4,n}/\Sigma_{n=1}{}^N p_{60,n}q_{A,4,n}]$. These partially chained Lowe indexes using annual weights lagged two years, $P_{LO3}{}^t$, are listed in Table 24 below and plotted in Chart 12.

Recall the notation for prices and quantities that was used in section 2 above: $q_{y,m,n}$ is the quantity of product n that is purchased by households in month m of year y and $p_{y,m,n}$ is the corresponding price where y = 1,...,Y; m = 1,...,M and n = 1,...,N.[60] *Annual expenditures for product n* purchased in year y, $e_{A,y,n}$, are obtained by summing expenditures on product n in year y over the months in the year; i.e., we have the following definitions:

(142) $e_{A,y,n} \equiv \Sigma_{m=1}{}^M p_{y,m,n}q_{y,m,n}$; $\qquad$ y = 1,...,Y; n = 1,...,N;
(143) $e_y \equiv \Sigma_{n=1}{}^N e_{A,y,m}$; $\qquad$ y = 1,...,Y;
(144) $s_{A,y,n} \equiv e_{A,y,n}/e_y$; $\qquad$ y = 1,...,Y; n = 1,...,N

where e_y is total expenditures on all products in year y and $s_{A,y,n}$ is the annual expenditure share of product n in year y. We will use the annual expenditure shares on products for year 1, $s_{A,1,n}$, in order to define our next index.

The *Fixed Base Young* (1812) *index* for month t using the annual weights of year 1, $P_{Y1}{}^t$, is defined as follows:

(145) $P_{Y1}{}^t \equiv \Sigma_{n=1}{}^N s_{A,1,n}(p_{t,n}/p_{1,n})$; $\qquad$ t = 1,...,T.

Using the data listed in Tables A23 and A24, this fixed base Young index $P_{Y1}{}^t$ is listed in Table 24 below and is plotted in Chart 12.[61]

The above Young index $P_{Y1}{}^t$ is not a real time Young index. Many statistical agencies use the annual expenditure share weights of the prior year and construct a real time Young index by using these lagged annual weights for one year and then they update their lagged annual weights for the following year. To approximate this type of index, we construct a (partially) *Chained Young index* for month t, $P_{Y2}{}^t$, as follows. For t = 1,...,24, define $P_{Y2}{}^t \equiv P_{Y1}{}^t$. Thus we use the annual expenditure shares for year 1 for the first 24 months of data. For a month t = 25,...,36 in the third year, define a new *link Young index* using the expenditure share weights of year 2 and the prices of year 3 relative to December of year 2, $\Sigma_{n=1}{}^N s_{A,2,n}(p_{t,n}/p_{24,n})$, and then link this index to the index

[60] Carry forward/backward prices are used for missing products in this section.
[61] The year 1 annual expenditure shares $s_{A,1,n}$ are as follows for our sample of 14 types of fruit: 0.07688, 0.09895, 0.12712, 0.04061, 0.00644, 0.18375, 0.14648, 0.07842, 0.03345, 0.02055, 0.05667, 0.07869, 0.01647, 0.03552.

value for P_{Y2}^{t} at t = 24. Thus for t = 25,...,36, define $P_{Y2}^{t} \equiv P_{Y2}^{24} \times \Sigma_{n=1}^{N} s_{A,2,n}(p_{t,n}/p_{24,n})$. For year 4, use the expenditure share weights of year 3 and the new Young link index that compares the prices of month t in year 4 to month 12 in year 3 to extend the definition of P_{Y2}^{t}. Thus for t = 37,...,48, define $P_{Y2}^{t} \equiv P_{Y2}^{36} \times \Sigma_{n=1}^{N} s_{A,3,n}(p_{t,n}/p_{36,n})$. In a similar manner, for t = 49,...,60, define $P_{Y2}^{t} \equiv P_{Y2}^{48} \times \Sigma_{n=1}^{N} s_{A,4,n}(p_{t,n}/p_{48,n})$ and for t = 61,...,72, define $P_{Y2}^{t} \equiv P_{Y2}^{60} \times \Sigma_{n=1}^{N} s_{A,5,n}(p_{t,n}/p_{60,n})$. These partially chained Young indexes using annual weights lagged one year, P_{Y2}^{t}, are listed in Table 24 below and plotted in Chart 12.

As was the case with Lowe indexes, some countries that produce Young indexes use annual expenditure share weights that are lagged two years. To approximate this type of index, we construct a *Lagged Two Year Chained Young index* for month t, P_{Y3}^{t}, as follows. For t = 1,...,36, define $P_{Y3}^{t} \equiv P_{Y1}^{t}$. Thus we use the annual expenditure shares for year 1 for the first 36 months of data to construct this alternative Young index which will be equal to our initial fixed base Young index, P_{Y1}^{t} , for the first three years of data. For the fourth year of data, define a new *link Young index* using the expenditure share weights of year 2 and the prices of year 4 relative to December of year 3, $\Sigma_{n=1}^{N} s_{A,2,n}(p_{t,n}/p_{36,n})$, and then link this index to the index value for P_{Y3}^{t} at t = 36. Thus for t = 37,...,48, define $P_{Y3}^{t} \equiv P_{Y3}^{36} \times \Sigma_{n=1}^{N} s_{A,2,n}(p_{t,n}/p_{36,n})$. In a similar manner, for t = 49,...,60, define $P_{Y3}^{t} \equiv P_{Y3}^{48} \times \Sigma_{n=1}^{N} s_{A,3,n}(p_{t,n}/p_{48,n})$ and for t = 61,...,72, define $P_{Y3}^{t} \equiv P_{Y3}^{60} \times \Sigma_{n=1}^{N} s_{A,4,n}(p_{t,n}/p_{60,n})$. These partially chained Young indexes using annual expenditure share weights lagged two years, P_{Y3}^{t}, are listed in Table 24 below and plotted in Chart 12.

For comparison purposes, the Maximum Overlap Predicted Share similarity linked indexes P_{S}^{t*} are also listed in Table 24 and plotted on Chart 12. These indexes were defined in section 7 above. Recall that these indexes had the "best" axiomatic and economic properties.

Table 24: Alternative Lowe and Young Indexes and Maximum Overlap Predicted Share Indexes

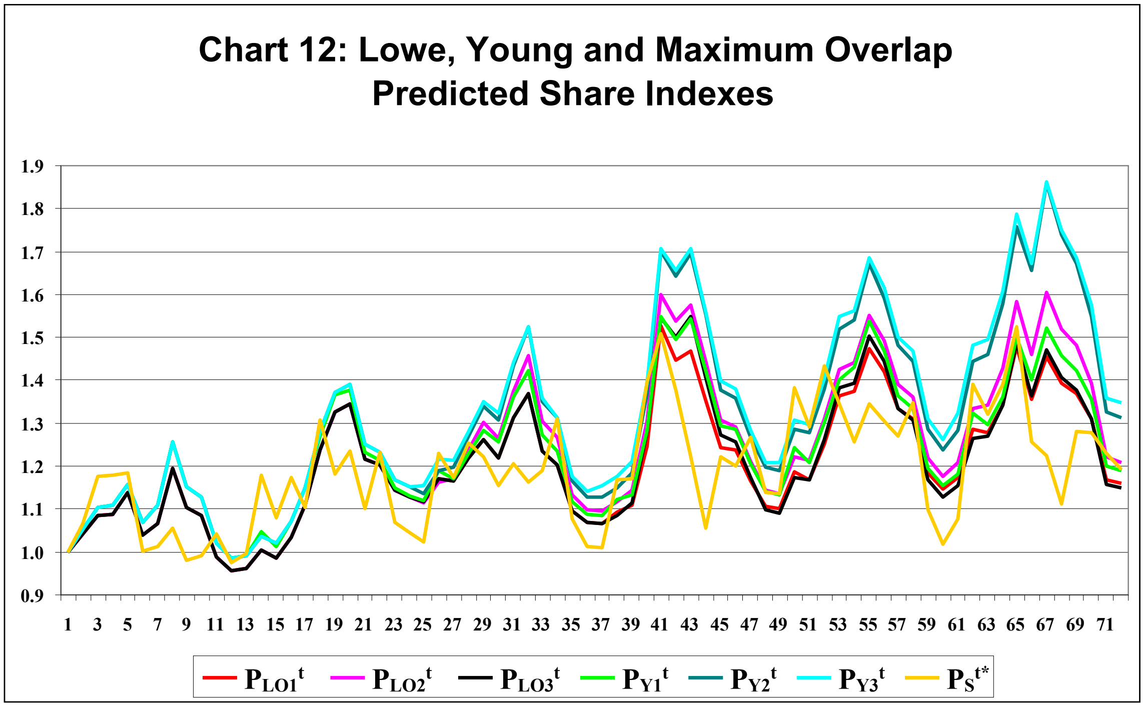
t	P_{LO1}^{t}	P_{LO2}^{t}	P_{LO3}^{t}	P_{Y1}^{t}	P_{Y2}^{t}	P_{Y3}^{t}	P_{S}^{t*}
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.04326	1.04326	1.04326	1.05268	1.05268	1.05268	1.06603
3	1.08548	1.08548	1.08548	1.10507	1.10507	1.10507	1.17647
4	1.08876	1.08876	1.08876	1.10788	1.10788	1.10788	1.17956
5	1.13936	1.13936	1.13936	1.15868	1.15868	1.15868	1.18310
6	1.03900	1.03900	1.03900	1.06992	1.06992	1.06992	1.00296
7	1.06729	1.06729	1.06729	1.10912	1.10912	1.10912	1.01198
8	1.19403	1.19403	1.19403	1.25591	1.25591	1.25591	1.05554
9	1.10376	1.10376	1.10376	1.15229	1.15229	1.15229	0.97973
10	1.08384	1.08384	1.08384	1.12758	1.12758	1.12758	0.99067
11	0.98815	0.98815	0.98815	1.02055	1.02055	1.02055	1.04107
12	0.95707	0.95707	0.95707	0.98609	0.98609	0.98609	0.97592
13	0.96145	0.96145	0.96145	0.99008	0.99144	0.99144	0.99684
14	1.00464	1.00464	1.00464	1.04855	1.03587	1.03587	1.17902
15	0.98637	0.98637	0.98637	1.01337	1.02120	1.02120	1.08056
16	1.03456	1.03456	1.03456	1.07048	1.07208	1.07208	1.17474
17	1.10996	1.10996	1.10996	1.15028	1.15016	1.15016	1.10498
18	1.23903	1.23903	1.23903	1.27051	1.27978	1.27978	1.30841
19	1.32716	1.32716	1.32716	1.36592	1.37230	1.37230	1.18142
20	1.34389	1.34389	1.34389	1.37751	1.39097	1.39097	1.23391
21	1.21519	1.21519	1.21519	1.23145	1.25047	1.25047	1.09986
22	1.20236	1.20236	1.20236	1.21261	1.23257	1.23257	1.23179
23	1.14380	1.14380	1.14380	1.14866	1.16844	1.16844	1.06906
24	1.12858	1.12858	1.12858	1.13117	1.15207	1.15207	1.04392
25	1.11756	1.11376	1.11756	1.11914	1.13660	1.15403	1.02270

26	**1.17002**	**1.16392**	**1.17002**	**1.19040**	**1.19004**	**1.21753**	**1.22856**
27	**1.16512**	**1.17153**	**1.16512**	**1.17183**	**1.19817**	**1.21480**	**1.17215**
28	**1.21869**	**1.23906**	**1.21869**	**1.22908**	**1.26991**	**1.28190**	**1.25327**
29	**1.26146**	**1.30297**	**1.26146**	**1.28205**	**1.34101**	**1.35073**	**1.22223**
30	**1.21967**	**1.26375**	**1.21967**	**1.25523**	**1.30879**	**1.32380**	**1.15449**
31	**1.31282**	**1.37439**	**1.31282**	**1.36083**	**1.43366**	**1.44250**	**1.20526**
32	**1.36885**	**1.45695**	**1.36885**	**1.42380**	**1.52351**	**1.52542**	**1.16278**
33	**1.23566**	**1.30444**	**1.23566**	**1.27315**	**1.35170**	**1.35787**	**1.18929**
34	**1.20359**	**1.26686**	**1.20359**	**1.23562**	**1.31157**	**1.31230**	**1.31066**
35	**1.09640**	**1.13257**	**1.09640**	**1.11793**	**1.16504**	**1.17490**	**1.07810**
36	**1.06780**	**1.09851**	**1.06780**	**1.08641**	**1.12818**	**1.13999**	**1.01195**
37	**1.06587**	**1.09628**	**1.06579**	**1.08420**	**1.12718**	**1.15462**	**1.01076**
38	**1.09187**	**1.11622**	**1.08513**	**1.12263**	**1.14963**	**1.17557**	**1.16812**
39	**1.11013**	**1.14497**	**1.11553**	**1.13424**	**1.18354**	**1.21131**	**1.17108**
40	**1.24637**	**1.30601**	**1.28028**	**1.28318**	**1.36839**	**1.39454**	**1.39663**
41	**1.52833**	**1.59991**	**1.54600**	**1.54905**	**1.70292**	**1.70586**	**1.50841**
42	**1.44653**	**1.53923**	**1.49936**	**1.49624**	**1.64162**	**1.65526**	**1.37756**
43	**1.46704**	**1.57449**	**1.55011**	**1.54226**	**1.69526**	**1.70704**	**1.22151**
44	**1.35281**	**1.44427**	**1.40749**	**1.42269**	**1.55393**	**1.56055**	**1.05506**
45	**1.24425**	**1.30625**	**1.27269**	**1.29453**	**1.37818**	**1.39849**	**1.22173**
46	**1.23748**	**1.29271**	**1.25707**	**1.28513**	**1.35919**	**1.38018**	**1.19999**
47	**1.16552**	**1.21047**	**1.17278**	**1.20599**	**1.26673**	**1.28470**	**1.26828**
48	**1.10676**	**1.14443**	**1.09805**	**1.14252**	**1.19705**	**1.20779**	**1.13921**
49	**1.10034**	**1.13626**	**1.09053**	**1.13441**	**1.18973**	**1.20886**	**1.13475**
50	**1.18691**	**1.22131**	**1.17300**	**1.24418**	**1.28716**	**1.30691**	**1.38339**
51	**1.16776**	**1.21323**	**1.16760**	**1.20785**	**1.27810**	**1.29792**	**1.29063**
52	**1.25206**	**1.30889**	**1.26315**	**1.30331**	**1.38020**	**1.40329**	**1.43303**
53	**1.36482**	**1.42634**	**1.38322**	**1.40161**	**1.52043**	**1.54805**	**1.34386**
54	**1.37422**	**1.44057**	**1.39343**	**1.43167**	**1.54143**	**1.56224**	**1.25757**
55	**1.47429**	**1.55230**	**1.50203**	**1.53753**	**1.67156**	**1.68595**	**1.34547**
56	**1.41979**	**1.49149**	**1.44386**	**1.46712**	**1.59174**	**1.61501**	**1.30412**
57	**1.33528**	**1.39119**	**1.33533**	**1.36367**	**1.48060**	**1.50056**	**1.26875**
58	**1.30946**	**1.36033**	**1.30726**	**1.33383**	**1.44496**	**1.46756**	**1.34737**
59	**1.18387**	**1.21857**	**1.16801**	**1.19556**	**1.28643**	**1.30911**	**1.09738**
60	**1.14666**	**1.17665**	**1.12767**	**1.15470**	**1.23912**	**1.26212**	**1.01922**
61	**1.17220**	**1.20921**	**1.15517**	**1.18209**	**1.28337**	**1.32271**	**1.07767**
62	**1.28637**	**1.33420**	**1.26463**	**1.32312**	**1.44306**	**1.48080**	**1.39115**
63	**1.27916**	**1.34273**	**1.26912**	**1.29807**	**1.46002**	**1.49454**	**1.32072**
64	**1.34213**	**1.42892**	**1.34161**	**1.36226**	**1.57761**	**1.60731**	**1.39001**
65	**1.47799**	**1.58492**	**1.48634**	**1.49954**	**1.75811**	**1.78672**	**1.52597**
66	**1.35495**	**1.46134**	**1.36328**	**1.40213**	**1.65693**	**1.67184**	**1.25740**
67	**1.45383**	**1.60427**	**1.47019**	**1.52241**	**1.85953**	**1.86366**	**1.22459**
68	**1.39418**	**1.52032**	**1.40637**	**1.45631**	**1.73987**	**1.74864**	**1.11160**
69	**1.36826**	**1.48091**	**1.37725**	**1.42296**	**1.67215**	**1.68685**	**1.27951**
70	**1.31139**	**1.39452**	**1.31040**	**1.35463**	**1.54773**	**1.57445**	**1.27885**
71	**1.16938**	**1.22085**	**1.15803**	**1.19906**	**1.32578**	**1.35928**	**1.23088**
72	**1.16123**	**1.20830**	**1.14936**	**1.18982**	**1.31397**	**1.34640**	**1.19115**
Mean	**1.20940**	**1.24830**	**1.21360**	**1.24240**	**1.31660**	**1.32920**	**1.18920**

The fixed base Young index $P_{Y1}{}^{t}$ ends up very close to our preferred index $P_S{}^{t*}$ at t = 72. However, the mean of the $P_{Y1}{}^{t}$ is 5.3 percentage points above the mean of the $P_S{}^{t*}$. It is interesting that the partially chained Young indexes, $P_{Y2}{}^{t}$ and $P_{Y3}{}^{t}$, appear to have some upward chain drift since they finished 12.3 and 15.5 percentage points above $P_S{}^{72*}$. Note that $P_{LO1}{}^{t}$, $P_{Y1}{}^{t}$ and $P_S{}^{t*}$ all satisfy the multiperiod identity test so these indexes are not subject to chain drift.[62] Turning to the Lowe

[62] However, as soon as the base period annual quantities or annual expenditure shares are updated, then the resulting Lowe and Young indexes will be subject to potential chain drift. The similarity linked indexes $P_S{}^{t*}$ always satisfy the multiperiod identity test and hence are not subject to chain drift.

indexes, the partially chained Lowe index that used annual quantity weights lagged one year, P_{LO2}^t, ended up 1.7 percentage points above where our preferred similarity linked index P_S^{t*} ended up. However, on average, P_{LO2}^t was 9.4 percentage points above P_S^{t*}. The fixed base Lowe index, P_{LO1}^t and the partially chained Lowe index that used annual quantity weights lagged two years, P_{LO3}^t, ended up 3.0 and 4.2 percentage points below P_S^{72*}.



From viewing Chart 12, it can be seen that none of the annual basket or annual expenditure share indexes provide an adequate approximation to our preferred similarity linked index, P_S^{t*}. The large upward seasonal fluctuations in the two partially chained Young indexes, P_{Y2}^t and P_{Y3}^t, are particular cause for concern. In general, the indexes that use annual quantities or expenditure shares as weights have an upward bias which is interesting since these indexes use carry forward prices so they should have a downward bias relative to P_S^{t*} since this similarity linked index does not use carry forward prices.

There is a conceptual problem with using annual basket indexes along with carry forward prices in the strongly seasonal commodities context. *The problem is that these indexes have no theoretical justification*. To see the problem clearly, think of an extreme case of strong seasonality for an elementary category where each commodity is available in only one month of the year. It is simply impossible to construct a meaningful price (or quantity) index for this category of goods or services. There is no basis for comparing the prices or quantities of one month or quarter with the corresponding prices or quantities of a different month or quarter of the same year since the product categories do not overlap.[63] In the strongly seasonal context where

[63] Andrew Baldwin (1990; 258) noted that there is a problem with using annual basket (AB) indexes in the seasonal context even if there is no strong seasonality: "For seasonal goods, the AB index is best considered an index partially adjusted for seasonal variation. It is based on annual quantities, which do not reflect the seasonal fluctuations in the volume of purchases, and on raw monthly prices, which do

there is some product overlap for months in the same year, we can construct meaningful price indexes between months in the same year for the set of overlap products between any two months and this is exactly what was done to create the similarity linked indexes P_S^{t*}.

Our conclusions for this section are as follows:

- In the strongly seasonal products context, Lowe and Young indexes using carry forward prices for missing products are subject to both carry forward bias and substitution bias and are unlikely to approximate alternative indexes that have better axiomatic and economic properties.
- Lowe and Young indexes have no rigorous conceptual foundation in the strongly seasonal context and do not provide answers to any practical price measurement problem.

It was noted above that Mudgett Stone indexes, which compare the prices of the current year with the prices of a previous year provide meaningful measures of annual inflation even in the case where each product in scope is only available in one month of the year. This type of index was studied in sections 4 and 5 above. In the following section, this type of index will be generalized to provide a measure of annual inflation that is updated each month. The resulting measures of price change can be compared to smoothed measures of month to month price change.

11. Rolling Year Measures of Annual Inflation and Measures of Trend Inflation

In sections 4 and 5 above, the price and quantity data pertaining to the 12 months of a calendar year were compared to the 12 months of a base calendar year. However, there is no need to restrict attention to calendar year comparisons: any 12 consecutive months of price and quantity data could be compared to the price and quantity data of the base year, provided that the January data in the noncalendar year is compared to the January data of the base year, the February data of the noncalendar year is compared to the February data of the base year, …, and the December data of the noncalendar year is compared to the December data of the base year.[64] Alterman, Diewert and Feenstra (1999; 70) called the resulting indexes *rolling year* indexes.[65] This approach to the measurement of price change is consistent with three of the four main approaches to index number theory: (i) the comparison of purchases of products in the two periods using either the base period consumption basket, the current period consumption basket or an average of the two[66]; (ii) the test approach and (iii) the stochastic approach.[67]

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incorporate seasonal price fluctuations. Zarnowitz (1961; 256-257) calls it an index of 'a hybrid sort'. Being neither of sea nor land, it does not provide an appropriate measure either of monthly or 12 month price change. The question that an AB index answers with respect to price change from January to February say, or January of one year to January of the next, is 'What would have the change in consumer prices have been if there were no seasonality in purchases in the months in question, but prices nonetheless retained their own seasonal behaviour?' It is hard to believe that this is a question that anyone would be interested in asking."

[64] Diewert (1983) suggested this type of comparison and termed the resulting index a "split year" comparison.

[65] Crump (1924; 185) used this term in the context of various seasonal adjustment procedures. Mendershausen (1937; 245) used the term "moving year". The term "rolling year" seems to be well established in the business literature in the UK.

[66] This leads to maximum overlap Laspeyres, Paasche and Fisher indexes.

[67] In order to rigorously justify rolling year indexes from the viewpoint of the economic approach to index number theory, some restrictions on preferences are required. The details of these assumptions can be found in Diewert (1999a; 56-61). The problems associated with forming annual indexes from monthly or

It is easy to explain how the rolling year indexes work in principle: the prices of the 12 months in the current rolling year are compared to the corresponding monthly prices in the base year, where January prices are matched up with January prices, February prices with February prices and so on. However, setting up the algebra for the maximum overlap Laspeyres and Paasche indexes is somewhat complex as will be seen below.

Recall that p^t and q^t are the month t vectors of dimension 14 for our example for t = 1,...,72. Treat these vectors as column vectors in what follows. The inner product of p^t and q^r is defined as $p^t \cdot q^r \equiv \Sigma_{n=1}^{14} p_{t,n}q_{r,n}$. If product n is not purchased in month t, then the nth components of p^t and q^t, $p_{t,n}$ and $q_{t,n}$, are set equal to 0. For t = 1,...,72, define the diagonal 14 by 14 matrix Δ^t as follows: if $q_{t,n} > 0$, then the element in the nth row and nth column of Δ^t is set equal to 1; if $q_{t,n} = 0$, then the element in the nth row and nth column of Δ^t is set equal to 0. The remaining components of Δ^t are set equal to 0.

The rolling year indexes cannot be defined until 12 months of data are available. Thus for our example data set which consists of 72 months of data, the rolling year indexes will run from t = 1 to t = 72. When t = 1, the first 12 months of data are compared with the first 12 months of data and the resulting rolling year index will equal 1. When t = 61, the rolling year index compares the last 12 months of data with the first 12 months of data.

The algebra for the *Rolling Year Fixed Base Maximum Overlap Laspeyres indexes* is set out below. This index for period t is denoted by $P_{LRY}^{t*} = \mathrm{Num}^t/\mathrm{Den}^t$. The numerators, Num^t, and the denominators, Den^t, for P_{LRY}^{t*} are defined as follows:

$\mathrm{Num}^1 \equiv \Sigma_{t=1}^{12} p^t \cdot q^t$; $\mathrm{Den}^1 \equiv \Sigma_{t=1}^{12} p^t \cdot q^t$;
$\mathrm{Num}^2 \equiv \mathrm{Num}^1 - p^1 \cdot q^1 + p^{13} \cdot q^1$; $\mathrm{Den}^2 \equiv \mathrm{Den}^1 - p^1 \cdot q^1 + p^1 \cdot \Delta^{13} q^1$;
$\mathrm{Num}^3 \equiv \mathrm{Num}^2 - p^2 \cdot q^2 + p^{14} \cdot q^2$; $\mathrm{Den}^3 \equiv \mathrm{Den}^2 - p^2 \cdot q^2 + p^2 \cdot \Delta^{14} q^2$;
$\mathrm{Num}^4 \equiv \mathrm{Num}^3 - p^3 \cdot q^3 + p^{15} \cdot q^3$; $\mathrm{Den}^4 \equiv \mathrm{Den}^3 - p^3 \cdot q^3 + p^3 \cdot \Delta^{15} q^3$;
●●● ●●●
$\mathrm{Num}^{13} \equiv \mathrm{Num}^{12} - p^{12} \cdot q^{12} + p^{24} \cdot q^{12}$; $\mathrm{Den}^{13} \equiv \mathrm{Den}^{12} - p^{12} \cdot q^{12} + p^{12} \cdot \Delta^{24} q^{12}$;
$\mathrm{Num}^{14} \equiv \mathrm{Num}^{13} - p^{13} \cdot q^1 + p^{25} \cdot q^1$; $\mathrm{Den}^{14} \equiv \mathrm{Den}^{13} - p^1 \cdot \Delta^{13} q^1 + p^1 \cdot \Delta^{25} q^1$;
$\mathrm{Num}^{15} \equiv \mathrm{Num}^{14} - p^{14} \cdot q^2 + p^{26} \cdot q^2$; $\mathrm{Den}^{15} \equiv \mathrm{Den}^{14} - p^2 \cdot \Delta^{14} q^2 + p^2 \cdot \Delta^{26} q^2$;
●●● ●●●
$\mathrm{Num}^{25} \equiv \mathrm{Num}^{24} - p^{24} \cdot q^{12} + p^{36} \cdot q^{12}$; $\mathrm{Den}^{25} \equiv \mathrm{Den}^{24} - p^{12} \cdot \Delta^{24} q^{12} + p^{12} \cdot \Delta^{36} q^{12}$;
$\mathrm{Num}^{26} \equiv \mathrm{Num}^{25} - p^{25} \cdot q^1 + p^{37} \cdot q^1$; $\mathrm{Den}^{26} \equiv \mathrm{Den}^{25} - p^1 \cdot \Delta^{25} q^1 + p^1 \cdot \Delta^{37} q^1$;
$\mathrm{Num}^{27} \equiv \mathrm{Num}^{26} - p^{26} \cdot q^2 + p^{38} \cdot q^2$; $\mathrm{Den}^{27} \equiv \mathrm{Den}^{26} - p^2 \cdot \Delta^{26} q^2 + p^2 \cdot \Delta^{38} q^2$;
●●● ●●●
$\mathrm{Num}^{37} \equiv \mathrm{Num}^{36} - p^{36} \cdot q^{12} + p^{48} \cdot q^{12}$; $\mathrm{Den}^{37} \equiv \mathrm{Den}^{36} - p^{12} \cdot \Delta^{36} q^{12} + p^{12} \cdot \Delta^{48} q^{12}$;
$\mathrm{Num}^{38} \equiv \mathrm{Num}^{37} - p^{37} \cdot q^1 + p^{49} \cdot q^1$; $\mathrm{Den}^{38} \equiv \mathrm{Den}^{37} - p^1 \cdot \Delta^{37} q^1 + p^1 \cdot \Delta^{49} q^1$;
$\mathrm{Num}^{39} \equiv \mathrm{Num}^{38} - p^{38} \cdot q^2 + p^{50} \cdot q^2$; $\mathrm{Den}^{39} \equiv \mathrm{Den}^{38} - p^2 \cdot \Delta^{38} q^2 + p^2 \cdot \Delta^{50} q^2$;
●●● ●●●
$\mathrm{Num}^{49} \equiv \mathrm{Num}^{48} - p^{48} \cdot q^{12} + p^{60} \cdot q^{12}$; $\mathrm{Den}^{49} \equiv \mathrm{Den}^{48} - p^{12} \cdot \Delta^{48} q^{12} + p^{12} \cdot \Delta^{60} q^{12}$;
$\mathrm{Num}^{50} \equiv \mathrm{Num}^{49} - p^{49} \cdot q^1 + p^{61} \cdot q^1$; $\mathrm{Den}^{50} \equiv \mathrm{Den}^{49} - p^1 \cdot \Delta^{49} q^1 + p^1 \cdot \Delta^{61} q^1$;
$\mathrm{Num}^{51} \equiv \mathrm{Num}^{50} - p^{50} \cdot q^2 + p^{62} \cdot q^2$; $\mathrm{Den}^{51} \equiv \mathrm{Den}^{50} - p^2 \cdot \Delta^{50} q^2 + p^2 \cdot \Delta^{62} q^2$;

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quarterly indexes have not been completely resolved from the viewpoint of the economic approach to index number theory.

•••

$Num^{61} \equiv Num^{60} - p^{60} \cdot q^{12} + p^{72} \cdot q^{12}$;

•••

$Den^{61} \equiv Den^{60} - p^{12} \cdot \Delta^{60} q^{12} + p^{12} \cdot \Delta^{72} q^{12}$;

The period t *Rolling Year* (fixed base maximum overlap) *Laspeyres Index* is defined as:

(146) $P_{LRY}^{t*} \equiv Num^t / Den^t$; $\quad t = 1,...,61.$

These Rolling Year Laspeyres indexes are listed in Table 25 below and plotted on Chart 13 below.

Recall that in section 5 above, the maximum overlap annual fixed base Laspeyres indexes, P_{LFB}^{y*} were defined by (69) for years y = 1,...,6. It can be verified that the Rolling Year Laspeyres indexes P_{LRY}^{t*} coincide with the earlier annual indexes P_{LFB}^{y*} for t = 1, 13, 25, 37, 49 and 61 and y = 1,...,6; i.e., we have $P_{LRY}^{1*} = P_{LFB}^{1*} = 1$, $P_{LRY}^{13*} = P_{LFB}^{2*}$, $P_{LRY}^{25*} = P_{LFB}^{3*}$, $P_{LRY}^{37*} = P_{LFB}^{4*}$, $P_{LRY}^{49*} = P_{LFB}^{5*}$ and $P_{LRY}^{61*} = P_{LFB}^{6*}$. Thus the new rolling year Laspeyres indexes defined in this section are a natural extension of the fixed base Mudgett Stone maximum overlap Laspeyres annual indexes defined in section 5. The new indexes provide a seasonally adjusted measure of annual inflation for the current split year that consists of the last consecutive 12 months relative to the corresponding seasonal prices prevailing in a base year.

The above algebra is modified to define the *Rolling Year Fixed Base Maximum Overlap Paasche indexes*, P_{PRY}^{t*}. The numerators, Num^t, and the denominators, Den^t, for P_{PRY}^{t*} are defined as follows:

$Num^1 \equiv \Sigma_{t=1}^{12} p^t \cdot q^t$;
$Num^2 \equiv Num^1 - p^1 \cdot q^1 + p^{13} \cdot \Delta^1 q^{13}$;
$Num^3 \equiv Num^2 - p^2 \cdot q^2 + p^{14} \cdot \Delta^2 q^{14}$;
$Num^4 \equiv Num^3 - p^3 \cdot q^3 + p^{15} \cdot \Delta^3 q^{15}$;
•••
$Num^{13} \equiv Num^{12} - p^{12} \cdot q^{12} + p^{24} \cdot \Delta^{12} q^{24}$;
$Num^{14} \equiv Num^{13} - p^{13} \cdot \Delta^1 q^{13} + p^{25} \cdot \Delta^1 q^{25}$;
$Num^{15} \equiv Num^{14} - p^{14} \cdot \Delta^2 q^{14} + p^{26} \cdot \Delta^2 q^{26}$;
•••
$Num^{25} \equiv Num^{24} - p^{24} \cdot \Delta^{12} q^{24} + p^{36} \cdot \Delta^{12} q^{36}$;
$Num^{26} \equiv Num^{25} - p^{25} \cdot \Delta^1 q^{25} + p^{37} \cdot \Delta^1 q^{37}$;
$Num^{27} \equiv Num^{26} - p^{26} \cdot \Delta^2 q^{26} + p^{38} \cdot \Delta^2 q^{38}$;
•••
$Num^{37} \equiv Num^{36} - p^{36} \cdot \Delta^{12} q^{36} + p^{48} \cdot \Delta^{12} q^{48}$;
$Num^{38} \equiv Num^{37} - p^{25} \cdot \Delta^1 q^{37} + p^{37} \cdot \Delta^1 q^{49}$;
$Num^{39} \equiv Num^{38} - p^{26} \cdot \Delta^2 q^{38} + p^{38} \cdot \Delta^2 q^{50}$;
•••
$Num^{49} \equiv Num^{48} - p^{48} \cdot \Delta^{12} q^{48} + p^{60} \cdot \Delta^{12} q^{60}$;
$Num^{50} \equiv Num^{49} - p^{49} \cdot \Delta^1 q^{49} + p^{61} \cdot \Delta^1 q^{61}$;
$Num^{51} \equiv Num^{50} - p^{50} \cdot \Delta^2 q^{50} + p^{61} \cdot \Delta^2 q^{62}$;
•••
$Num^{61} \equiv Num^{60} - p^{60} \cdot \Delta^{12} q^{60} + p^{72} \cdot \Delta^{12} q^{72}$;

$Den^1 \equiv \Sigma_{t=1}^{12} p^t \cdot q^t$;
$Den^2 \equiv Den^1 - p^1 \cdot q^1 + p^1 \cdot q^{13}$;
$Den^3 \equiv Den^2 - p^2 \cdot q^2 + p^2 \cdot q^{14}$;
$Den^4 \equiv Den^3 - p^3 \cdot q^3 + p^3 \cdot q^{15}$;
•••
$Den^{13} \equiv Den^{12} - p^{12} \cdot q^{12} + p^{12} \cdot q^{24}$;
$Den^{14} \equiv Den^{13} - p^1 \cdot q^{13} + p^1 \cdot q^{25}$;
$Den^{15} \equiv Den^{14} - p^2 \cdot q^{14} + p^2 \cdot q^{26}$;
•••
$Den^{25} \equiv Den^{24} - p^{12} \cdot q^{24} + p^{12} \cdot q^{36}$;
$Den^{26} \equiv Den^{25} - p^1 \cdot q^{25} + p^1 \cdot q^{37}$;
$Den^{27} \equiv Den^{26} - p^2 \cdot q^{26} + p^2 \cdot q^{38}$;
•••
$Den^{37} \equiv Den^{36} - p^{12} \cdot q^{36} + p^{12} \cdot q^{48}$;
$Den^{38} \equiv Den^{37} - p^1 \cdot q^{37} + p^1 \cdot q^{49}$;
$Den^{39} \equiv Den^{38} - p^2 \cdot q^{38} + p^2 \cdot q^{50}$;
•••
$Den^{49} \equiv Den^{48} - p^{12} \cdot q^{48} + p^{12} \cdot q^{60}$;
$Den^{50} \equiv Den^{49} - p^1 \cdot q^{49} + p^1 \cdot q^{61}$;
$Den^{51} \equiv Den^{50} - p^2 \cdot q^{50} + p^2 \cdot q^{62}$;
•••
$Den^{61} \equiv Den^{60} - p^{12} \cdot q^{60} + p^{12} \cdot q^{72}$.

The period t *Rolling Year* (fixed base maximum overlap) *Paasche Index* is defined as:

(147) $P_{PRY}^{t*} \equiv Num^t / Den^t$; $\quad t = 1,...,61.$

The period t *Rolling Year* (fixed base maximum overlap) *Fisher Index*, $P_{FRY}{}^{t*}$ is defined as the geometric mean of the period t Rolling year Laspeyres and Paasche indexes:

(148) $P_{FRY}{}^{t*} \equiv (P_{LRY}{}^{t*}\, P_{PRY}{}^{t*})^{1/2}$; $t = 1,...,61.$

These Rolling Year Paasche and Fisher indexes are listed in Table 25 below and plotted on Chart 13 below.[68]

Table 25: Rolling Year Maximum Overlap Laspeyres, Paasche and Fisher Price Indexes and Some Moving Average Approximations

t	$P_{LRY}{}^{t*}$	$P_{PRY}{}^{t*}$	$P_{FRY}{}^{t*}$	$P_{SMA}{}^{t*}$	$P_{FMMA}{}^{t*}$	$P_{SMMA}{}^{t*}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.99986	0.99993	0.99990	0.99975	0.99984	0.99984
3	1.01275	1.00477	1.00875	1.00867	1.00831	1.00831
4	1.00672	0.99713	1.00192	1.00110	1.00158	1.00158
5	1.00599	0.99599	1.00097	1.00072	1.00037	1.00037
6	1.00217	0.98795	0.99504	0.99455	0.99487	0.99487
7	1.05211	1.01266	1.03220	1.01867	1.02025	1.02025
8	1.09356	1.03034	1.06148	1.03205	1.04268	1.04268
9	1.11096	1.04550	1.07774	1.04614	1.05676	1.05676
10	1.12245	1.05347	1.08741	1.05562	1.06698	1.06698
11	1.12983	1.05505	1.09180	1.07467	1.07090	1.07090
12	1.13288	1.05680	1.09418	1.07688	1.07315	1.07315
13	1.13733	1.06109	1.09855	1.08225	1.07893	1.07893
14	1.13936	1.06215	1.10008	1.08429	1.08124	1.08099
15	1.14418	1.06902	1.10596	1.08820	1.08665	1.08485
16	1.15089	1.07688	1.11327	1.09543	1.09337	1.09134
17	1.15589	1.08247	1.11858	1.10163	1.09979	1.09776
18	1.16438	1.09490	1.12910	1.11089	1.10894	1.10602
19	1.14176	1.08425	1.11264	1.09874	1.09814	1.09323
20	1.11904	1.07086	1.09469	1.10062	1.08460	1.08040
21	1.11661	1.06762	1.09184	1.09500	1.08185	1.07840
22	1.12505	1.07248	1.09845	1.10207	1.08856	1.08601
23	1.12847	1.07606	1.10195	1.10829	1.09327	1.09149
24	1.12908	1.07732	1.10289	1.10901	1.09458	1.09221
25	1.12732	1.07502	1.10086	1.10648	1.09196	1.08956
26	1.12612	1.07412	1.09981	1.10554	1.09071	1.08856
27	1.12030	1.06836	1.09403	1.10077	1.08528	1.08385
28	1.11999	1.06776	1.09356	1.10068	1.08496	1.08377
29	1.12900	1.07624	1.10231	1.11200	1.09596	1.09390
30	1.15203	1.09503	1.12317	1.13460	1.11745	1.11406
31	1.18096	1.11449	1.14724	1.15222	1.13439	1.13259
32	1.18007	1.11904	1.14915	1.15350	1.13552	1.13385
33	1.16794	1.10638	1.13674	1.14500	1.12513	1.12501
34	1.16815	1.10961	1.13850	1.14756	1.12787	1.12777
35	1.17315	1.12032	1.14643	1.13882	1.13684	1.13597
36	1.18452	1.13220	1.15807	1.15384	1.15148	1.15118
37	1.19194	1.14068	1.16603	1.16389	1.16170	1.16144
38	1.19810	1.14830	1.17294	1.17368	1.17126	1.17173
39	1.21756	1.16467	1.19082	1.19068	1.18805	1.18849
40	1.22668	1.17024	1.19813	1.20012	1.19509	1.19647
41	1.22795	1.17032	1.19879	1.20299	1.19598	1.19801
42	1.21506	1.16311	1.18880	1.19000	1.18545	1.18642

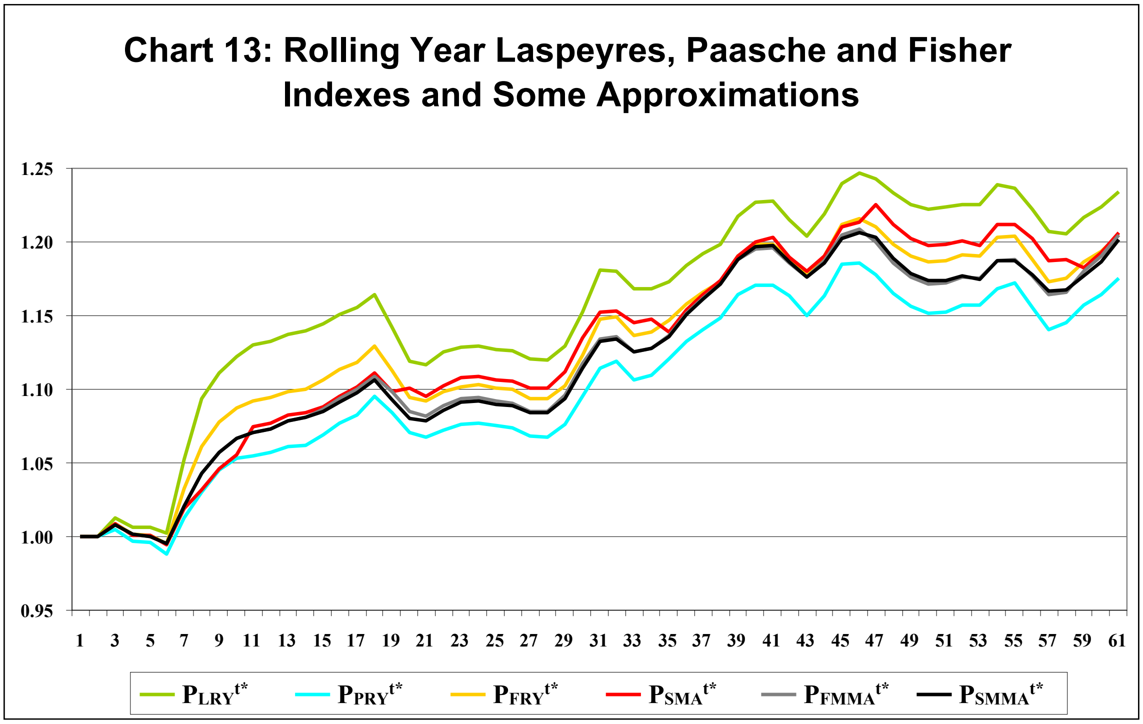
[68] The indexes defined in the last three columns of Table 25 will be defined later.

43	**1.20435**	**1.14967**	**1.17670**	**1.18052**	**1.17641**	**1.17645**
44	**1.21910**	**1.16329**	**1.19087**	**1.19031**	**1.18707**	**1.18601**
45	**1.23970**	**1.18457**	**1.21182**	**1.20998**	**1.20506**	**1.20239**
46	**1.24684**	**1.18607**	**1.21608**	**1.21369**	**1.20894**	**1.20639**
47	**1.24310**	**1.17770**	**1.20996**	**1.22533**	**1.20009**	**1.20336**
48	**1.23297**	**1.16529**	**1.19866**	**1.21184**	**1.18601**	**1.18911**
49	**1.22527**	**1.15652**	**1.19040**	**1.20236**	**1.17608**	**1.17882**
50	**1.22248**	**1.15126**	**1.18633**	**1.19785**	**1.17143**	**1.17343**
51	**1.22365**	**1.15275**	**1.18767**	**1.19847**	**1.17242**	**1.17404**
52	**1.22568**	**1.15747**	**1.19108**	**1.20084**	**1.17603**	**1.17667**
53	**1.22548**	**1.15675**	**1.19062**	**1.19745**	**1.17531**	**1.17467**
54	**1.23906**	**1.16849**	**1.20326**	**1.21183**	**1.18741**	**1.18749**
55	**1.23623**	**1.17260**	**1.20399**	**1.21181**	**1.18786**	**1.18748**
56	**1.22222**	**1.15521**	**1.18824**	**1.20227**	**1.17682**	**1.17816**
57	**1.20681**	**1.14061**	**1.17325**	**1.18706**	**1.16390**	**1.16641**
58	**1.20534**	**1.14562**	**1.17510**	**1.18791**	**1.16585**	**1.16733**
59	**1.21683**	**1.15703**	**1.18655**	**1.18250**	**1.18044**	**1.17700**
60	**1.22373**	**1.16429**	**1.19364**	**1.19305**	**1.19014**	**1.18686**
61	**1.23439**	**1.17520**	**1.20443**	**1.20662**	**1.20447**	**1.20137**
Mean	**1.15940**	**1.10380**	**1.13120**	**1.13060**	**1.12120**	**1.12050**

Using the means listed in Table 25, it can be seen that the Rolling Year Laspeyres indexes, $P_{LRY}{}^{t*}$, are on average 2.8 percentage points above the corresponding Rolling Year Fisher indexes, $P_{FRY}{}^{t*}$, while the Rolling Year Paasche indexes, $P_{PRY}{}^{t*}$, are on average 2.7 percentage points below the corresponding Rolling Year Fisher indexes. This indicates that the Laspeyres and Paasche indexes suffer from a considerable amount of substitution bias.

It can be verified that the Rolling Year Laspeyres, Paasche and Fisher indexes, $P_{LRY}{}^{t*}$, $P_{PRY}{}^{t*}$ and $P_{FRY}{}^{t*}$, coincide with the corresponding annual indexes listed in Table 12 in section 5, $P_{LFB}{}^{y*}$, $P_{PFB}{}^{y*}$ and $P_{FFB}{}^{y*}$, for t = 1, 13, 25, 37, 49, 61 and y = 1,...,6.

The indexes listed in Table 25 above are plotted in Chart 13.



A comparison of the Laspeyres, Paasche and Fisher Rolling Year indexes shown on Chart 13 with the month to month indexes that are plotted on Charts 7 to 12 indicates that the Rolling Year indexes are much less variable than any of the month to month indexes. Thus the Rolling Year indexes capture both the trend in inflation as well as eliminating the seasonal fluctuations in the month to month measures of inflation. The upward bias in the Rolling Year Laspeyres index and the downward bias in the Rolling Year Paasche index are apparent in Chart 13.

At this point, our conclusions in this section are as follows:

- Rolling Year Maximum Overlap Laspeyres, Paasche and Fisher indexes can readily be calculated provided monthly information on prices and quantities is available.
- These indexes are a natural generalization of the Annual Mudgett Stone indexes defined in section 5 above to provide annual index numbers for non-calendar years. They have the advantage that they can provide a new measure of trend inflation each month; i.e., one does not have to wait until the end of a calendar year to get a current measure of inflation.
- These indexes can be regarded as seasonally adjusted measures of trend inflation that is centered in the middle of the current non-calendar year that consists of the last string of 12 consecutive months. In the strongly seasonal context, these indexes provide the most accurate measures of inflation.
- As usual, the Rolling Year Maximum Overlap Fisher index of annualized inflation is preferred over the counterpart Rolling Year Laspeyres and Paasche indexes which suffer from substitution bias.

In addition to the Rolling Year Fixed Base Laspeyres, Paasche and Fisher indexes that are listed in Table 25 and are plotted on Chart 13, there are 3 additional indexes that are listed in Table 25.

These additional indexes are approximations to other indexes and hence are not of primary importance but they are of interest.

The first additional index of interest is a moving average of our "best" month to month maximum overlap similarity linked indexes P_S^{t*} defined in section 9. How well can such an index approximate the Rolling Year Fisher index P_{FRY}^{t*} defined above? The 12 month moving average of the P_S^{t*}, P_{MA}^t, is defined as follows:

$$(149)\ P_{MA}^1 \equiv (1/12)\Sigma_{m=1}^{12} P_S^{m*}\ ;\ P_{MA}^t \equiv P_{MA}^{t-1} + (1/12)P_S^{t+11*} - (1/12)P_S^{t-1*}\ ; \qquad t = 2,3,...,61.$$

To make a price index out of the above series of moving averages , divide the P_{MA}^t by P_{MA}^1. Thus the *smoothed version of the month to month similarity linked indexes* P_S^{t*} is the 12 month moving average series P_{SMA}^{t*} defined as follows;

$$(150)\ P_{SMA}^{t*} \equiv P_{MA}^t/P_{MA}^1\ ; \qquad t = 1,2,...,61.$$

The smoothed month to month similarity indexes P_{SMA}^{t*} represent estimates of the *trend* in the month to month relative price similarity linked indexes P_S^{t*}. Thus P_{SMA}^1 represents the trend in $P_S^{1*} - P_S^{12*}$ centered in the middle of year 1 of our sample; P_{SMA}^2 represents the trend in $P_S^{2*} - P_S^{13*}$ centered in the middle of the split year consisting of months 2-12 in year 1 and January in year 2 and so on. Table 25 and Chart 13 shows that the trend indexes P_{SMA}^{t*} are fairly close to the Rolling Year Fixed Base Maximum Overlap Fisher indexes P_{FRY}^{t*}; the two indexes end up at 1.2066 and 1.2044 respectively and their means are 1.1306 and 1.1312 respectively. Thus for our particular data set, the Rolling Year Fisher indexes not only have an explicit annual index number interpretation, but they also provide an estimate for the trend in the month to month similarity linked Fisher indexes, P_S^{t*}.[69]

We conclude this section by describing the last two additional indexes of interest that appear on Chart 13.

In section 4, we saw that the true Mudgett Stone annual Laspeyres index could be computed as a share weighted average of the monthly year over year indexes. In section 5, we took a simple equally weighted average of the maximum overlap fixed base year over year monthly Fisher indexes $P_{FFB}^{y,m*}$ and showed that the resulting index could provide an approximation to the "true" Annual Mudgett Stone Fixed Base Fisher indexes P_{FFB}^{y*}. The resulting annual Mudgett Stone indexes were defined by (82) and denoted by P_{FFBA}^{y*} for y = 1,...,Y. The same type of approximation can be made for the Rolling Year Fisher indexes, P_{FRY}^{t*}. We indicate how these approximate rolling year Fisher indexes, P_{FMMA}^{t*} can be defined.

$P_{FFB}^{y,m*}$ is the year over year monthly fixed base maximum overlap Fisher price index for month m in year y. These indexes were defined in section 3 above and are listed in Table A22 in the Appendix. We simplify the notation and define P(y,m) as follows:

[69] However, it should be kept in mind that the similarity linked month to month indexes P_S^{t*} are conceptually quite different from the Rolling Year Fisher indexes P_{FRY}^{t*}. In the case where all products are strongly seasonal and appear in only one month of the year, the Rolling Year Mudgett Stone indexes are still well defined and meaningful from an economic perspective whereas month to month indexes maximum overlap indexes cannot even be defined in this case. For a review of the early history of time series methods for measuring trends and providing seasonally adjusted series, see Diewert, Alterman and Feenstra (2012). Oskar Anderson (1927; 552-554) provided a very clear statement of the arbitrariness of existing methods for decomposing time series into trend, seasonal and erratic components.

(151) $P(y,m) \equiv P_{FFB}{}^{y,m*}$; $\qquad$ $y = 1,...,6;\ m = 1,...,12.$

The 12 month moving averages of these indexes, P^t for $t = 1,...,61$, are defined as follows:

$P^1 \equiv (1/12)\Sigma_{m=1}{}^{12} P(1,m)$
$P^2 \equiv P^1 + (1/12)P(2,1) - (1/12)P(1,1)$
$P^3 \equiv P^2 + (1/12)P(2,2) - (1/12)P(1,2)$
$P^4 \equiv P^3 + (1/12)P(2,3) - (1/12)P(1,3)$
●●●
$P^{13} \equiv P^{12} + (1/12)P(2,12) - (1/12)P(1,12)$
$P^{14} \equiv P^{13} + (1/12)P(3,1) - (1/12)P(2,1)$
$P^{15} \equiv P^{14} + (1/12)P(3,2) - (1/12)P(2,2)$
●●●
$P^{25} \equiv P^{24} + (1/12)P(3,12) - (1/12)P(2,12)$
$P^{26} \equiv P^{25} + (1/12)P(4,1) - (1/12)P(3,1)$
$P^{27} \equiv P^{24} + (1/12)P(4,2) - (1/12)P(3,2)$
●●●
$P^{37} \equiv P^{36} + (1/12)P(4,12) - (1/12)P(3,12)$
$P^{38} \equiv P^{36} + (1/12)P(5,1) - (1/12)P(4,1)$
$P^{39} \equiv P^{37} + (1/12)P(5,2) - (1/12)P(4,2)$
●●●
$P^{49} \equiv P^{48} + (1/12)P(5,12) - (1/12)P(4,12)$
$P^{50} \equiv P^{49} + (1/12)P(6,1) - (1/12)P(5,1)$
$P^{51} \equiv P^{50} + (1/12)P(6,2) - (1/12)P(5,2)$
●●●
$P^{61} \equiv P^{60} + (1/12)P(6,12) - (1/12)P(5,12).$

Normalize the above 12 month moving averages into an index which equals 1 in the base period. Define the *Moving Average Index of the Year over Year Monthly Fixed Base Maximum Overlap Fisher indexes*, $P_{FMMA}{}^{t*}$, as follows:

(152) $P_{FMMA}{}^{t*} \equiv P^t/P^1$; $\qquad$ $t = 1,...,61$

where the P^t are defined by (151). The $P_{FMMA}{}^{t*}$ are listed in Table 25 and are plotted in Chart 13.

Instead of using the year over year monthly fixed base maximum overlap Fisher indexes as basic building blocks to form the approximate Rolling Year index $P_{FMMA}{}^{t*}$, other year over year indexes could be used as basic monthly building blocks, such as the maximum overlap similarity linked monthly year over year monthly indexes $P_S{}^{y,m*}$ defined below Table 6 in section 3. These indexes are also listed in Table A22 in the Appendix. To construct the resulting approximate Similarity Linked Rolling Year index $P_{SMMA}{}^{t*}$, redefine $P(y,m)$ as follows:

(153) $P(y,m) \equiv P_S{}^{y,m*}$; $\qquad$ $y = 1,...,6;\ m = 1,...,12.$

The 12 month moving averages of these indexes, P^t for $t = 1,...,61$, can be defined using the algebra listed below (151) above but using definitions (153) for $P(y,m)$. Define the *Moving Average Index of the Year over Year Monthly Similarity Linked indexes*, $P_{SMMA}{}^{t*}$, as follows:

(154) $P_{SMMA}{}^{t*} \equiv P^t/P^1$; $\qquad$ t = 1,...,61

where the P^t are defined by the algebra below (151) . The $P_{SMMA}{}^{t*}$ are listed in Table 25 and are plotted in Chart 13.

Viewing Table 25 and Chart 13, it can be seen that the indexes $P_{FMMA}{}^{t*}$ and $P_{SMMA}{}^{t*}$ (which are normalized 12 month moving average series of the Fisher Fixed Base and Similarity Linked Maximum Overlap Year over Year Monthly indexes $P_{FFB}{}^{y,m*}$ and $P_S{}^{y,m*}$) closely approximate each other and can barely be distinguished in Chart 13. This is to be expected since the underlying year over year monthly series, $P_{FFB}{}^{y,m*}$ and $P_S{}^{y,m*}$, closely approximate each other.

In section 5, two approximate annual maximum overlap Mudgett Stone indexes, $P_{FFB}{}^{y*}$ and $P_{SA}{}^{y*}$, were defined and listed in Table 13 above for y = 1,...,6. The indexes $P_{FMMA}{}^{t*}$ and $P_{SMMA}{}^{t*}$ are extensions of these indexes to rolling years. Thus we have $P_{FFB}{}^{2*} = P_{FMMA}{}^{13*}$, $P_{FFB}{}^{3*} = P_{FMMA}{}^{25*}$, $P_{FFB}{}^{4*} = P_{FMMA}{}^{37*}$, $P_{FFB}{}^{5*} = P_{FMMA}{}^{49*}$ and $P_{FFB}{}^{6*} = P_{FMMA}{}^{61*}$. Similarly, comparing entries in Tables 13 and 25, we have $P_{SA}{}^{2*} = P_{SMMA}{}^{13*}$, $P_{SA}{}^{3*} = P_{SMMA}{}^{25*}$, $P_{SA}{}^{4*} = P_{SMMA}{}^{37*}$, $P_{SA}{}^{5*} = P_{SMMA}{}^{49*}$ and $P_{SA}{}^{6*} = P_{SMMA}{}^{61*}$. Thus the indexes $P_{FMMA}{}^{t*}$ and $P_{SMMA}{}^{t*}$ are natural extensions of the approximate calendar year annual Mudgett Stone indexes $P_{FFB}{}^{y*}$ and $P_{SA}{}^{y*}$ to split years.

Our preferred rolling year index is the Rolling Year maximum overlap fixed base Fisher index, $P_{FRY}{}^{t*}$. This index and the two approximate indexes, $P_{FMMA}{}^{t*}$ and $P_{SMMA}{}^{t*}$ ended up at 1.20433, 1.20447 and 1.20137 respectively for our empirical example, which is more or less the same place. However, the means of the three indexes were 1.1312, 1.1212 and 1.1205 respectively. Thus the two approximate indexes were on average about 1 percentage point below the mean of the Fisher Rolling Year index, $P_{FRY}{}^{t*}$. The two approximate Rolling year indexes capture the trend quite well but they give equal weights to each of the 12 months in the Rolling Year and thus are not as accurate (from the viewpoint of the economic approach to index number theory) as the Rolling Year Fisher index which weights the 12 year over year monthly index according to their economic importance.

Viewing Table 25 and Chart 13, it can be seen that the indexes $P_{FMMA}{}^{t*}$ and $P_{SMMA}{}^{t*}$ (which are normalized 12 month moving average series of the Fisher Fixed Base and Similarity Linked Maximum Overlap Year over Year Monthly indexes $P_{FFB}{}^{y,m*}$ and $P_S{}^{y,m*}$) closely approximate each other and can barely be distinguished in Chart 13. This is to be expected since the underlying year over year monthly series, $P_{FFB}{}^{y,m*}$ and $P_S{}^{y,m*}$, closely approximate each other.

12. Conclusion

The existence of strongly seasonal commodities raises a large number of problems that national statistical offices face when attempting to construct Consumer Price Indexes that include strongly seasonal product categories.

This chapter has considered *four main classes of alternative price indexes* that could be constructed for a strongly seasonal class of commodities:

- Year over year monthly indexes (see sections 2 and 3 above);
- Annual indexes (see sections 4,5 and 11);
- Month to month indexes that measure consumer price inflation going from one month to the next month (see sections 6 and 7 for indexes that make use of price and quantity information and sections 8 and 9 for indexes that use only price information);

- Month to month annual basket indexes (or annual share indexes) that make use of annual quantities or annual expenditure shares for a base year and monthly prices (see section 10 for the Lowe and Young indexes).

As was discussed in section 10, in the strongly seasonal commodities context, Lowe or Young indexes have little intuitive appeal. Consumers do not purchase an annual basket of strongly seasonal commodities in each month nor do they face carry forward prices each month for this hypothetical annual basket of commodities.

The other three types of index have strong justifications. Month to month indexes are required by central banks and others to monitor short run movements in inflation. Annual indexes are needed as deflators to produce annual constant dollar national accounts. Strictly speaking, year over year monthly indexes do not have as high a priority as month to month and annual indexes but it turns out that in the strongly seasonal commodities context, year over year monthly indexes are far more accurate measures of inflation than month to month indexes. Moreover, the year over year monthly indexes are basic building blocks for accurate annual indexes. *Thus in the strongly seasonal commodities context, all three types of index serve a useful purpose*. This is our first important conclusion.

There are five other more technical issues that proved to be important in producing price indexes for strongly seasonal commodity groups:

- *Should carry forward/backward prices be used for missing prices in constructing a price index* or should maximum overlap indexes be produced (which is roughly equivalent to using inflation adjusted carry forward prices for missing prices)? Common sense and our computations show that the use of carry forward prices in the strongly seasonal context will lead to a downward bias in the index if there is general inflation (and vice versa if there is general deflation as has occurred in Japan at times). Thus the use of carry forward prices is not recommended.
- *Are monthly price and quantity (or expenditure) data available or are just monthly price data available*? The type of index that can be produced depends on data availability. Of course, indexes that make use of price and quantity information are preferred but statistical offices usually do not have price and quantity information so the issue of which index to use in the prices only situation is important. Our results in section 9 show that, for our empirical example, it is possible to come up with a prices only index that can provide a fairly satisfactory approximation to our "best" index that makes use of price and quantity information.
- *What is the "best" bilateral index number formula to use when making price comparisons between two periods*? When price and quantity information are available for the two periods under consideration, the Fisher price index is a good candidate for the "best" index. It has very good properties from the perspectives of both the economic approach to index number theory[70] as well as the test approach.[71] When only price information is available, the choice of a "best" functional form for a bilateral index is not so clear. If there are no missing prices (or if prices are completely matched across the two periods), then the Jevons index has the best axiomatic properties.[72] In

[70] See Diewert (1976).
[71] See Diewert (1992; 221).
[72] See Diewert (1995). The economic approach to index number theory that relies on exact index number formulae cannot be implemented if only price information is available..

the case where where prices are not matched across the two periods, the best approach at this stage of our knowledge is probably the maximum overlap Jevons index.

- However, the choice of a "best" bilateral index number formula is not the end of the story. In making index comparisons across multiple time periods using bilateral indexes as basic building blocks to link the prices of any pair of periods, one has to choose a path of bilateral links in order to link all of the periods. For example, one can choose the first period as the base period and link all subsequent periods to this base period, generating a sequence of fixed base indexes. Or one can calculate a chained index where the prices of period t are linked to the prices of period t − 1 and this chain link index is used to update the period t index level. The problem with fixed base indexes in the strongly seasonal context when producing month to month indexes is that the choice of base period matters to a very significant degree; see Chart 5 in section 6 and Chart 7 in section 7. The problem with chained indexes is that they are subject to *chain drift*; i.e., if prices are identical in any two periods, it is desirable that the price index register the same index level for those two periods. Fixed base indexes will satisfy this test but chained indexes will in general not satisfy this multiperiod identity test. There are numerous examples in this chapter that show that chain drift can be a very significant problem when one uses chained indexes. *Thus there is the problem of choosing a "best" path to link bilateral price indexes into a single index*. Our suggested solution to this problem is to use a *measure of relative price dissimilarity* between the prices of any two periods and choose a path of bilateral links that minimizes the measure of price dissimilarity between the prices of the current period and the prices of all previous periods (up to some specified limit on how far back we want to go with the bilateral relative price comparisons). The price dissimilarity measure determines the path of bilateral links. If price and quantity information is available, then bilateral maximum overlap Fisher indexes are used to make the bilateral links in the chosen path. If only price information is available, then maximum overlap Jevons indexes are used to make the bilateral links. The resulting indexes satisfy the multiperiod identity test and hence are free from chain drift. The main problem with this methodology is this: what is the "best" dissimilarity measure that could be used? We do not provide a definitive answer to this question but *the Predicted Share measure of relative price dissimilarity* suggested by Diewert (2021b) seems to work well for our empirical example when price and quantity information is available. When only price information is available, we adapted the Predicted Share measure of relative price dissimilarity to deal with this case; see definitions (131) and (139) in section 9 above. For our particular example, this *Modified Predicted Share* method (P_{SJ}^{t*}) that used maximum overlap Jevons indexes for the bilateral links provided the closest approximation to our preferred Predicted Share similarity linked indexes (P_S^{t*}) that used price and quantity information and maximum overlap Fisher indexes for the bilateral links.[73]
- *How to trade off a lack of matching of prices over two periods with a lack of price proportionality in the matched prices for the two periods*? In section 9, we showed how the Modified Predicted Share measure of relative price dissimilarity traded off a lack of matching of product prices over the two periods under consideration with a measure of relative price dissimilarity of the matched prices for the two periods. In the strongly seasonal commodities context, it is important to have a penalty for a lack of matching of prices between the two periods being compared. Consider an extreme case where we are matching the prices of a current period with the prices of two prior periods. For period 1, there is only one matched product so if we look at only matched product

[73] See Chart 11 in section 9.

> prices, the matched prices of period 1 and 3 are proportional and any reasonable measure of relative price dissimilarity defined over matched prices will register a value of zero. On the other hand, there are 10 matched prices for periods 2 and 3 but the resulting matched prices are not quite proportional so the measure of relative price dissimilarity over matched products registers a positive value. Is it "best" to link the prices of period 3 with the single price of period 1 rather than link the prices of period 3 with the prices of period 2? Probably not. Thus we think it is important for a bilateral measure of relative price dissimilarity to have a penalty for a lack of matching of prices between the two periods. The Predicted Share and Modified Predicted Share measures of relative price dissimilarity do have a penalty for a lack of matching. Further research is required to see if "better" measures of relative price dissimilarity can be found.[74]

Another area that requires further research is the problem of integrating an elementary index for a strongly seasonal class of commodities with indexes for other elementary categories where the problems associated with missing prices are not as severe. Thus different elementary categories of a national CPI may use different methods for constructing the various subindexes. As a result, it may become difficult to explain and interpret the resulting national index.

For our data set, the year over year monthly indexes (January data compared across years, February data compared across years and so on) performed well. Thus for National Statistical Offices that are forced to use Lowe type indexes for various reasons, we suggest the use of monthly baskets for strongly seasonal commodities so that reasonably accurate year over year monthly Lowe indexes could be computed. However, the problem of how to link these indexes for a base year so that the year over year indexes could be aligned to provide some indication of January to February inflation, February to March inflation and so on for a base year. This could be done for the base year using some form of relative price similarity linking or one could choose a base month in the base year which had the highest number of available products and use fixed base maximum overlap Lowe or Fisher indexes to link the months in the base year. Then going forward, year over year monthly Lowe type indexes could be used to calculate the index.[75] Our general advice to National Statistical Officies is to move to the use of monthly baskets and similarity linking, particulary in current times when Covid induced lockdowns and preference changes have caused substantial fluctuations in monthly consumer expenditure shares in many countries.[76]

Finally, we note that we have listed the complete data set that we used in the Appendix so that our results can be replicated by statistical agencies.[77] Moreover, this data set could be used by other researchers to construct alternative indexes which may turn out to have superior properties.[78]

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[74] Research on this topic is sparse but see Hill and Timmer (2006) for an alternative approach to these issues.

[75] The problems associated with reconciling the year over year estimates of inflation for each month with month to month estimates of inflation within a given year are similar to the problems associated with reconciling year over year annual CPI country inflation estimates with estimates of inflation across countries for the same year. The annual CPI inflation rates for a given country are very likely to be much more accurate than a measure of relative inflation across countries due to better matching of commodity prices within a country, which is analogous to the better matching of product prices across years for the same month in the strongly seasonal context. For a discussion of alternative approaches to reconciling the conflicting estimates of inflation, see Diewert and Fox (2017) (2018).

[76] Statistics Canada has used the Predicted Share linking methodology in its Adjusted Consumer Price Index; see O' Donnell and Yélou (2021).

[77] We have also taken care to carefully explain exactly how the various indexes listed in this chapter were constructed.

Appendix: Listing of the Data and Supplementary Tables.

1. Year over Year Monthly Indexes Using Year over Year Carry Forward Prices

In order to illustrate the variation in the various seasonal commodity indexes using actual country data, we table the various indexes described in the main text for Israel for 14 fresh fruit household consumption categories over the 6 years 2012-2017 which we relabel as years 1-6. The 14 fresh fruit categories are the following ones:

- 1 = Lemons
- 2 = Avocados
- 3 = Watermelon
- 4 = Persimmon
- 5 = Grapefruit
- 6 = Bananas
- 7 = Peaches
- 8 = Strawberries
- 9 = Cherries
- 10 = Apricots
- 11 = Plums
- 12 = Clementines
- 13 = Kiwi fruit
- 14 = Mangos.

The price and quantity data for the available products in each month are listed below in Tables A.1-A.20. The price and quantity for product n in month m in year t are denoted by $p_{y,m,n}$ and $q_{y,m,n}$ respectively.

Table A.1: Year over Year Price and Quantity Data for Month 1 (January)

y	$p_{y,1,1}$	$p_{y,1,2}$	$p_{y,1,4}$	$p_{y,1,5}$	$p_{y,1,6}$	$p_{y,1,12}$	$p_{y,1,13}$	$q_{y,1,1}$	$q_{y,1,2}$	$q_{y,1,4}$	$q_{y,1,5}$	$q_{y,1,6}$	$q_{y,1,12}$	$q_{y,1,13}$
1	5.41	8.29	9.46	4.88	6.22	5.81	11.82	0.370	0.676	0.465	0.082	1.881	1.824	0.135
2	6.28	8.70	10.55	5.21	5.57	5.89	10.72	0.430	0.897	0.417	0.058	1.957	1.579	0.103
3	6.63	8.88	10.49	5.03	5.44	6.30	14.94	0.513	0.890	0.486	0.298	2.261	1.175	0.067
4	6.20	8.00	8.94	4.99	6.27	5.83	14.98	0.645	0.975	0.559	0.160	2.281	1.492	0.100
5	7.07	11.13	12.59	5.35	6.12	5.93	14.59	0.552	1.006	0.485	0.093	2.647	1.737	0.206
6	6.51	9.64	11.11	5.25	6.07	5.83	17.88	0.906	1.172	0.630	0.057	3.262	2.093	0.056

Fruits 1, 2, 4, 5, 6, 12 and 13 were always available in January for each of the six years in our sample; the other fruits were always missing in January. The price and quantity data for February and the remaining months follow below. Prices and quantities for products that are missing in a given month for all 6 years are not listed in the tables.

Table A.2: Year over Year Price Data for Month 2 (February)

y	$p_{y,2,1}$	$p_{y,2,2}$	$p_{y,2,4}$	$p_{y,2,5}$	$p_{y,2,6}$	$p_{y,2,8}$	$p_{y,2,12}$	$p_{y,2,13}$
1	4.99	8.37	10.20	4.84	6.90	15.08	6.32	12.78

[78] Turvey's (1979) artificial data set on seasonal commodities filled this role for many years.

2	**5.93**	**9.15**	**11.41**	**5.21**	**5.57**	**23.29**	**6.43**	**11.58**
3	**5.97**	**8.84**	**11.32**	**5.03**	**5.98**	**25.11**	**6.50**	**14.92**
4	**5.97**	**8.15**	**9.95**	**5.14**	**6.06**	**23.49**	**5.94**	**15.41**
5	**6.99**	**12.27**	**13.22**	**5.09**	**7.22**	**26.86**	**6.15**	**14.88**
6	**6.39**	**10.59**	**11.85**	**5.00**	**8.23**	**28.26**	**5.65**	**18.97**

Table A.3: Year over Year Quantity Data for Month 2 (February)

y	$q_{y,2,1}$	$q_{y,2,2}$	$q_{y,2,4}$	$q_{y,2,5}$	$q_{y,2,6}$	$q_{y,2,8}$	$q_{y,2,12}$	$q_{y,2,13}$
1	**0.701**	**0.920**	**0.510**	**0.103**	**2.087**	**1.134**	**1.408**	**0.102**
2	**0.624**	**0.831**	**0.412**	**0.269**	**2.621**	**0.593**	**1.664**	**0.155**
3	**0.754**	**1.075**	**0.486**	**0.119**	**2.308**	**0.737**	**1.492**	**0.168**
4	**0.553**	**1.031**	**0.412**	**0.156**	**2.591**	**0.766**	**1.919**	**0.117**
5	**0.658**	**0.717**	**0.386**	**0.157**	**2.299**	**0.648**	**1.886**	**0.108**
6	**0.657**	**0.859**	**0.447**	**0.260**	**2.211**	**0.711**	**2.071**	**0.111**

Fruits 1, 2, 4, 5, 6, 8, 12 and 13 were always available in February for each of the six years in our sample; the other fruits were always missing in February.

Table A.4: Year over Year Price Data for Month 3 (March)

y	$p_{y,3,1}$	$p_{y,3,2}$	$p_{y,3,4}$	$p_{y,3,5}$	$p_{y,3,6}$	$p_{y,3,8}$	$p_{y,3,12}$	$p_{y,3,13}$
1	**5.14**	**8.59**	**10.76**	**4.92**	**7.42**	**18.67**	**6.62**	**13.34**
2	**5.70**	**9.43**	**11.69**	**5.16**	**6.11**	**15.31**	**6.64**	**11.72**
3	**5.72**	**9.47**	**12.41**	**4.97**	**6.51**	**18.23**	**6.82**	**15.36**
4	**6.08**	**9.06**	**11.02**	**4.98**	**6.83**	**18.95**	**6.17**	**15.73**
5	**6.78**	**13.98**	*11.02*	**5.13**	**7.51**	**18.06**	**6.03**	**15.11**
6	**6.32**	**11.05**	**13.66**	**5.24**	**8.85**	**19.26**	**6.06**	**19.66**

Table A.5: Year over Year Quantity Data for Month 3 (March)

y	$q_{y,3,1}$	$q_{y,3,2}$	$q_{y,3,4}$	$q_{y,3,5}$	$q_{y,3,6}$	$q_{y,3,8}$	$q_{y,3,12}$	$q_{y,3,13}$
1	**0.661**	**0.908**	**0.362**	**0.081**	**1.819**	**0.884**	**1.269**	**0.112**
2	**0.684**	**0.732**	**0.257**	**0.116**	**2.242**	**0.947**	**1.160**	**0.154**
3	**0.822**	**0.612**	**0.290**	**0.121**	**2.012**	**0.845**	**1.496**	**0.085**
4	**0.658**	**0.828**	**0.209**	**0.181**	**2.255**	**0.813**	**1.556**	**0.108**
5	**0.708**	**0.694**	**0.000**	**0.234**	**2.490**	**1.107**	**1.509**	**0.152**
6	**0.759**	**0.787**	**0.293**	**0.095**	**2.395**	**1.038**	**1.650**	**0.102**

Note that product 4 is missing in March of year 5; i.e., $q_{5,3,4} = 0$. The corresponding price, $p_{5,3,4} = 11.02$, is an *imputed carry forward price* from March of year 4. In Table A.4, this imputed price is printed in *italics* to distinguish it from observed prices. Products 1, 2, 5, 6, 8, 12 and 13 were present in every April. Products 3, 7, 9, 10 and 11 were missing in every March.

Table A.6: Year over Year Price and Quantity Data for Month 4 (April)

y	$p_{y,4,1}$	$p_{y,4,2}$	$p_{y,4,5}$	$p_{y,4,6}$	$p_{y,4,8}$	$p_{y,4,12}$	$p_{y,4,13}$	$q_{y,4,1}$	$q_{y,4,2}$	$q_{y,4,5}$	$q_{y,4,6}$	$q_{y,4,8}$	$q_{y,4,12}$	$q_{y,4,13}$
1	**5.08**	**9.06**	**5.13**	**7.25**	**18.24**	**7.01**	**13.70**	**0.689**	**0.585**	**0.156**	**1.876**	**0.609**	**0.728**	**0.131**
2	**6.84**	**11.00**	**5.43**	**6.26**	**16.62**	**7.18**	**12.42**	**0.760**	**0.591**	**0.092**	**2.141**	**0.698**	**0.766**	**0.129**
3	**6.00**	**10.27**	**5.09**	**7.60**	**17.80**	**7.72**	**16.91**	**0.617**	**0.662**	**0.157**	**1.737**	**0.663**	**0.997**	**0.053**
4	**7.04**	**12.60**	**5.41**	**9.68**	**18.35**	**7.03**	**16.30**	**0.895**	**0.683**	**0.129**	**1.550**	**0.687**	**1.252**	**0.135**
5	**7.05**	**18.26**	**5.07**	**8.40**	**18.80**	**6.58**	**16.36**	**0.766**	**0.460**	**0.079**	**1.988**	**0.585**	**1.231**	**0.122**
6	**6.47**	**12.59**	**5.45**	**10.75**	**16.85**	**6.28**	**20.39**	**0.773**	**0.627**	**0.037**	**2.047**	**0.926**	**1.210**	**0.069**

Fruits 1, 2, 5, 6, 8, 12 and 13 were always available in April for each of the six years in our sample; the other fruits were always missing in April.

Table A.7: Year over Year Price Data for Month 5 (May)

y	$p_{y,5,1}$	$p_{y,5,2}$	$p_{y,5,3}$	$p_{y,5,5}$	$p_{y,5,6}$	$p_{y,5,7}$	$p_{y,5,8}$	$p_{y,5,9}$	$p_{y,5,10}$	$p_{y,5,13}$
1	5.19	11.48	4.14	5.27	7.05	11.50	16.68	*40.84*	12.16	13.69
2	7.35	14.62	3.49	5.67	5.96	11.08	*16.68*	*40.84*	9.46	*13.69*
3	6.60	13.66	4.10	5.34	7.60	10.62	*16.68*	40.84	14.79	19.93
4	7.73	15.92	4.56	5.39	13.19	11.75	*16.68*	61.43	17.78	17.16
5	7.52	19.36	4.07	5.81	8.98	11.27	*16.68*	39.10	18.31	17.33
6	7.00	15.34	4.77	6.16	12.30	12.95	*16.68*	39.10	18.03	22.56

Table A.8: Year over Year Quantity Data for Month 5 (May)

y	$q_{y,5,1}$	$q_{y,5,2}$	$q_{y,5,3}$	$q_{y,5,5}$	$q_{y,5,6}$	$q_{y,5,7}$	$q_{y,5,8}$	$q_{y,5,9}$	$q_{y,5,10}$	$q_{y,5,13}$
1	0.751	0.409	4.106	0.076	1.730	0.922	0.456	0.000	0.206	0.080
2	0.626	0.321	6.504	0.053	1.913	1.273	0.000	0.000	0.370	0.000
3	0.682	0.417	5.244	0.075	1.526	1.525	0.000	0.088	0.176	0.045
4	0.660	0.528	4.211	0.056	1.054	1.183	0.000	0.016	0.107	0.041
5	0.785	0.584	5.430	0.103	1.726	1.287	0.000	0.138	0.284	0.069
6	0.814	0.587	5.891	0.065	1.504	1.243	0.000	0.000	0.322	0.040

Products 1, 2, 3, 5, 6 and 10 are always present in May. However, product 8 is only present in year 1 of our sample so the price for product 8 in year 1 is *carried forward* for years 2-6; thus $p_{y,5,8}$ is set equal to $p_{y,5,8} = 16.68$ for y = 2,3,4,5,6. Product 9 is missing in years 1 and 2 and so the price for product 9 in years 1 and 2 is set equal to the *carry backward* price for product 9 in year 3; i.e., $p_{1,5,9}$ and $p_{2,5,9}$ are set equal to $p_{3,5,9} = 40.84$, the price of product 9 in year 3. The price of product 13 is missing in year 2 so this missing price is set equal to the price of product 13 in year 1; i.e., we have $p_{2,5,13} = p_{1,5,13.}$ thus Table A.7 has 8 imputed prices (which are in italics): 6 imputed carry forward prices and 2 imputed carry backward prices. Products 4, 11, 12 and 14 are missing in May for every year in our sample.

Table A.9: Year over Year Price Data for Month 6 (June)

y	$p_{y,6,1}$	$p_{y,6,2}$	$p_{y,6,3}$	$p_{y,6,5}$	$p_{y,6,6}$	$p_{y,6,7}$	$p_{y,6,9}$	$p_{y,6,10}$	$p_{y,6,11}$	$p_{y,6,13}$
1	5.66	11.83	3.24	5.57	5.92	10.08	17.44	8.82	11.05	13.74
2	7.83	19.58	3.36	5.86	5.85	11.25	42.05	14.44	12.61	*13.74*
3	6.64	14.00	2.55	5.63	8.07	10.42	32.81	13.25	14.03	27.25
4	8.62	18.98	3.68	*5.63*	12.71	10.80	34.48	12.36	13.56	21.55
5	9.01	20.42	2.67	*5.63*	10.99	9.73	34.21	15.05	13.62	22.38
6	8.20	18.56	2.93	*5.63*	11.15	9.81	31.08	14.71	14.01	26.03

Table A.10: Year over Year Quantity Data for Month 6 (June)

y	$q_{y,6,1}$	$q_{y,6,2}$	$q_{y,6,3}$	$q_{y,6,5}$	$q_{y,6,6}$	$q_{y,6,7}$	$q_{y,6,9}$	$q_{y,6,10}$	$q_{y,6,11}$	$q_{y,6,13}$
1	0.724	0.440	6.698	0.036	1.486	1.657	0.717	1.270	0.290	0.022
2	0.766	0.266	7.738	0.051	1.419	1.831	0.228	0.616	0.523	0.000
3	0.678	0.450	8.118	0.036	1.016	1.910	0.466	0.694	0.335	0.011
4	0.673	0.295	7.038	0.000	0.653	1.880	0.299	0.777	0.354	0.014
5	0.599	0.318	8.876	0.000	0.792	1.922	0.406	0.472	0.382	0.031
6	0.915	0.436	9.693	0.000	0.969	2.487	0.560	0.727	0.378	0.019

There are 4 missing prices for the products that appear for one or more months in June. Product 5 is missing in years 4, 5 and 6 and product 13 is missing in year 2. These 4 missing prices are replaced by carry forward prices (in italics) in Table A.9. Products 4, 8, 12 and 14 are always missing in June.

Table A.11: Year over Year Price Data for Month 7 (July)

y	$p_{y,7,1}$	$p_{y,7,2}$	$p_{y,7,3}$	$p_{y,7,6}$	$p_{y,7,7}$	$p_{y,7,9}$	$p_{y,7,11}$	$p_{y,7,14}$
1	7.40	13.02	3.18	6.65	9.27	20.10	8.82	10.41
2	9.96	*13.02*	3.30	7.46	11.91	53.77	10.65	10.92
3	7.51	15.44	2.29	10.76	11.23	37.98	12.59	10.63
4	9.83	*15.44*	2.51	15.91	10.24	30.62	10.36	12.32
5	11.34	*15.44*	3.09	12.56	10.66	37.31	12.85	11.35
6	10.86	*15.44*	2.32	14.74	9.87	35.14	11.23	13.48

Table A.12: Year over Year Quantity Data for Month 7 (July)

y	$q_{y,7,1}$	$q_{y,7,2}$	$q_{y,7,3}$	$q_{y,7,6}$	$q_{y,7,7}$	$q_{y,7,9}$	$q_{y,7,11}$	$q_{y,7,14}$
1	0.595	0.292	8.145	0.722	2.093	0.488	0.964	0.221
2	0.612	0.000	7.394	0.871	1.520	0.073	0.761	0.421
3	0.746	0.389	9.869	0.539	1.915	0.179	0.667	0.546
4	0.600	0.000	8.486	0.289	2.129	0.349	0.685	0.390
5	0.635	0.000	8.188	0.701	2.073	0.198	0.545	0.643
6	0.847	0.000	12.845	0.468	2.837	0.361	0.784	0.593

Product 2 is missing in years 2, 4, 5 and 6 so carry forward prices (in italics) appear for these 4 prices in Table A.11. Fruits 1, 2, 3, 6, 7, 9, 11 and 14 appear in at least one July; the remaining 6 fruits are not available in July.

Table A.13: Year over Year Price Data for Month 8 (August)

y	$p_{y,8,1}$	$p_{y,8,2}$	$p_{y,8,3}$	$p_{y,8,6}$	$p_{y,8,7}$	$p_{y,8,9}$	$p_{y,8,11}$	$p_{y,8,14}$
1	10.62	18.23	3.28	8.24	9.06	*22.50*	8.13	9.15
2	9.44	*18.23*	3.83	7.78	11.53	*22.50*	10.94	10.35
3	8.23	19.44	3.12	10.56	11.84	22.50	13.30	8.94
4	9.87	*19.44*	2.51	12.25	10.14	*22.50*	9.61	10.40
5	10.30	*19.44*	4.01	9.65	10.73	*22.50*	13.20	11.19
6	10.87	*19.44*	2.60	12.20	10.39	*22.50*	11.09	11.37

Table A.14: Year over Year Quantity Data for Month 8 (August)

y	$q_{y,8,1}$	$q_{y,8,2}$	$q_{y,8,3}$	$q_{y,8,6}$	$q_{y,8,7}$	$q_{y,8,9}$	$q_{y,8,11}$	$q_{y,8,14}$
1	0.452	0.159	6.159	0.558	1.932	0.000	1.009	0.721
2	0.625	0.000	5.065	0.746	1.761	0.000	0.914	0.850
3	0.656	0.180	5.577	0.616	1.791	0.031	0.759	1.040
4	0.719	0.000	7.371	0.498	1.765	0.000	0.832	0.673
5	0.718	0.000	4.963	0.974	2.171	0.000	0.750	1.028
6	0.690	0.000	8.423	0.770	2.348	0.000	0.748	0.730

Product 2 is missing in years 2, 4, 5 and 6 so carry forward prices (in italics) appear for these 4 prices in Table A.11. Product 9 is missing for years 1 and 2 (use carry backward prices) and years 4, 5 and 6 (use carry forward prices). Thus there are 9 missing prices for the August data. Fruits 1, 2, 3, 6, 7, 9, 11 and 14 appear in at least one August; the remaining 6 fruits are not available in August.

Table A.15: Year over Year Price and Quantity Data for Month 9 (September)

y	$p_{y,9,1}$	$p_{y,9,2}$	$p_{y,9,6}$	$p_{y,9,7}$	$p_{y,9,11}$	$p_{y,9,12}$	$p_{y,9,14}$	$q_{y,9,1}$	$q_{y,9,2}$	$q_{y,9,6}$	$q_{y,9,7}$	$q_{y,9,11}$	$q_{y,9,12}$	$q_{y,9,14}$
1	9.27	11.65	8.55	8.03	8.15	6.88	10.03	0.647	0.335	0.643	2.379	1.104	0.058	0.857
2	8.00	11.92	7.21	10.07	11.31	7.36	10.85	0.650	0.235	0.957	1.927	0.716	0.231	0.710
3	7.12	12.03	9.34	11.43	13.44	7.20	9.66	0.758	0.532	0.921	1.899	0.543	0.181	0.932
4	9.42	12.23	9.59	10.85	11.18	7.89	12.29	0.594	0.278	0.792	1.604	0.689	0.114	0.667
5	8.91	13.52	8.39	11.77	14.61	7.40	11.52	0.831	0.473	1.335	2.022	0.568	0.243	0.972
6	10.11	17.74	10.52	10.72	12.03	7.98	12.49	0.752	0.282	1.502	2.136	0.948	0.188	0.945

Fruits 1, 2, 6, 7 11, 12 and 14 are present in every September for the 6 years in our sample. The remaining 7 products are absent in all September months.

Table A.16: Year over Year Price Data for Month 10 (October)

y	$p_{y,10,1}$	$p_{y,10,2}$	$p_{y,10,3}$	$p_{y,10,5}$	$p_{y,10,6}$	$p_{y,10,7}$	$p_{y,10,9}$	$p_{y,10,10}$	$p_{y,10,11}$	$p_{y,10,13}$
1	8.15	11.26	11.45	6.59	7.93	9.18	8.18	6.19	14.63	9.87
2	8.03	9.8	12.3	6.43	6.8	*9.18*	12.32	7.19	15.47	13.09
3	7.14	10.51	12.98	6.21	8.79	*9.18*	14.5	7.33	17.43	9.67
4	9.53	11.54	12.85	7.28	8.91	11.94	11.58	7.22	20.19	11.88
5	8.35	12.65	13.61	6.57	7.88	*11.94*	*11.58*	7.07	22.85	12.87
6	9.62	14.86	13.85	6.81	8.92	12.67	13.13	7.09	21.74	12.49

Table A.17: Year over Year Quantity Data for Month 10 (October)

y	$q_{y,10,1}$	$q_{y,10,2}$	$q_{y,10,3}$	$q_{y,10,5}$	$q_{y,10,6}$	$q_{y,10,7}$	$q_{y,10,9}$	$q_{y,10,10}$	$q_{y,10,11}$	$q_{y,10,13}$
1	0.724	0.409	0.428	0.030	1.021	1.569	0.648	0.420	0.055	0.395
2	0.635	0.673	0.537	0.078	1.721	0.000	0.373	0.612	0.045	0.306
3	0.742	0.666	0.108	0.048	1.365	0.000	0.269	0.641	0.040	0.486
4	0.724	0.537	0.117	0.041	1.459	1.508	0.717	0.402	0.064	0.438
5	1.018	0.735	0.272	0.046	2.183	0.000	0.000	0.863	0.101	0.420
6	0.811	0.505	0.159	0.044	1.996	1.294	0.457	0.367	0.055	0.456

Product 7 is missing in years 2, 3 and 5; product 11 is missing in year 5. These 4 missing prices are replaced by carry forward prices. Products 3, 8, 9 and 10 are missing in every October.

Table A.18: Year over Year Price Data for Month 11 (November)

y	$p_{y,11,1}$	$p_{y,11,2}$	$p_{y,11,4}$	$p_{y,11,5}$	$p_{y,11,6}$	$p_{y,11,11}$	$p_{y,11,12}$	$p_{y,11,13}$	$p_{y,11,14}$
1	7.30	8.89	9.80	5.96	6.09	8.78	5.96	10.45	10.32
2	7.46	8.90	9.82	5.89	6.07	*8.78*	6.43	14.12	*10.32*
3	6.73	8.58	9.80	5.78	6.40	*8.78*	6.62	14.98	12.26
4	8.82	10.13	12.40	6.27	7.75	11.68	6.62	15.13	12.26
5	7.26	9.58	11.31	6.00	5.89	*11.68*	6.17	19.60	*12.26*
6	8.49	11.51	11.06	6.95	6.57	12.93	6.34	17.18	*12.26*

Table A.19: Year over Year Quantity Data for Month 11 (November)

y	$q_{y,11,1}$	$q_{y,11,2}$	$q_{y,11,4}$	$q_{y,11,5}$	$q_{y,11,6}$	$q_{y,11,11}$	$q_{y,11,12}$	$q_{y,11,13}$	$q_{y,11,14}$
1	0.712	0.765	0.510	0.101	2.069	0.410	1.309	0.124	0.223
2	0.603	1.000	0.601	0.085	2.405	0.000	1.726	0.064	0.000
3	0.594	0.897	0.510	0.087	2.328	0.000	1.344	0.080	0.220
4	0.612	1.066	0.435	0.080	2.400	0.283	1.148	0.099	0.000
5	0.992	1.075	0.557	0.150	2.920	0.000	1.313	0.087	0.000
6	0.836	0.990	0.443	0.158	3.014	0.116	1.167	0.076	0.179

Product 11 is missing in years 2, 3 and 5; product 14 is missing in years 2, 4 and 5. These 6 missing prices are replaced by carry forward prices. Products 3, 7, 8, 9 and 10 are missing in every November.

Table A.20: Year over Year Price and Quantity Data for Month 12 (December)

y	$p_{y,12,1}$	$p_{y,12,2}$	$p_{y,12,4}$	$p_{y,12,5}$	$p_{y,12,6}$	$p_{y,12,12}$	$p_{y,12,13}$	$q_{y,12,1}$	$q_{y,12,2}$	$q_{y,12,4}$	$q_{y,12,5}$	$q_{y,12,6}$	$q_{y,12,12}$	$q_{y,12,13}$
1	5.41	8.29	9.46	4.88	6.22	5.81	11.82	0.370	0.676	0.465	0.082	1.881	1.824	0.135
2	6.28	8.70	10.55	5.21	5.57	5.89	10.72	0.430	0.897	0.417	0.058	1.957	1.579	0.103
3	6.63	8.88	10.49	5.03	5.44	6.30	14.94	0.513	0.890	0.486	0.298	2.261	1.175	0.067
4	6.20	8.00	8.94	4.99	6.27	5.83	14.98	0.645	0.975	0.559	0.160	2.281	1.492	0.100
5	7.07	11.13	12.59	5.35	6.12	5.93	14.59	0.552	1.006	0.485	0.093	2.647	1.737	0.206
6	6.51	9.64	11.11	5.25	6.07	5.83	17.88	0.906	1.172	0.630	0.057	3.262	2.093	0.056

Fruits 1, 2, 4, 5, 6, 12 and 13 were always available in December for each of the six years in our sample; the remaining 7 fruits were always missing in December.

Over all 12 months, there were 34 missing prices that were imputed. 30 of the imputed prices were carry forward prices and 4 of the imputed prices were carry backward prices.

The above data series were used to compute all of the year over year monthly indexes that are listed in the following Table:

Table A.21: Year over Year Indexes for Months Using Carry Forward Prices

y	m	$P_{LFB}^{y,m}$	$P_{PFB}^{y,m}$	$P_{FFB}^{y,m}$	$P_{TFB}^{y,m}$	$P_{LCH}^{y,m}$	$P_{PCH}^{y,m}$	$P_{FCH}^{y,m}$	$P_{TCH}^{y,m}$	$P_{GEKS}^{y,m}$	$P_S^{y,m}$
1	1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1	0.99746	0.99881	0.99813	0.99817	0.99746	0.99881	0.99813	0.99817	0.99814	0.99813
3	1	1.03276	1.01894	1.02583	1.02591	1.02762	1.01799	1.02280	1.02261	1.02295	1.02280
4	1	1.01159	1.00992	1.01076	1.01072	1.01586	0.99872	1.00725	1.00700	1.00816	1.01076
5	1	1.12212	1.12896	1.12554	1.12582	1.14808	1.10989	1.12883	1.12854	1.12973	1.13415
6	1	1.07410	1.06543	1.06976	1.06889	1.09958	1.04827	1.07362	1.07252	1.07153	1.06944
1	2	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	2	1.14673	1.05830	1.10163	1.09970	1.14673	1.05830	1.10163	1.09970	1.10937	1.10163
3	2	1.19856	1.13544	1.16657	1.16430	1.19530	1.10240	1.14791	1.14597	1.15856	1.14791
4	2	1.13489	1.06908	1.10149	1.09983	1.13690	1.04779	1.09144	1.08957	1.10156	1.09144
5	2	1.35079	1.25687	1.30298	1.30238	1.35316	1.23472	1.29259	1.29006	1.30486	1.29259
6	2	1.36333	1.26804	1.31482	1.31429	1.36508	1.23771	1.29984	1.29727	1.31271	1.29984
1	3	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	3	0.92742	0.91116	0.91925	0.91910	0.92742	0.91116	0.91925	0.91910	0.91727	0.91925
3	3	1.00396	0.99578	0.99986	0.99981	1.00995	0.98455	0.99717	0.99686	0.99912	0.99717
4	3	1.00033	0.99176	0.99603	0.99611	1.00911	0.98358	0.99626	0.99588	0.99714	0.99626
5	3	1.09322	1.06264	1.07782	1.07646	1.09945	1.05318	1.07607	1.07519	1.07794	1.08539
6	3	1.13016	1.11723	1.12368	1.12351	1.15073	1.10109	1.12564	1.12495	1.12558	1.12368
1	4	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	4	0.98803	0.98284	0.98543	0.98569	0.98803	0.98284	0.98543	0.98569	0.98766	0.98543
3	4	1.06459	1.06038	1.06248	1.06235	1.06796	1.04900	1.05844	1.05817	1.06550	1.06248
4	4	1.20496	1.18402	1.19444	1.19482	1.19860	1.16073	1.17951	1.17928	1.19142	1.18402
5	4	1.22481	1.18576	1.20513	1.20454	1.23398	1.15532	1.19400	1.19293	1.20392	1.20245
6	4	1.22173	1.17182	1.19652	1.19732	1.24896	1.13499	1.19061	1.18951	1.19466	1.17841
1	5	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	5	0.95731	0.91814	0.93752	0.93708	0.95731	0.91814	0.93752	0.93708	0.93879	0.93752
3	5	1.04955	1.02931	1.03938	1.03929	1.07750	0.99674	1.03634	1.03544	1.04223	1.03938
4	5	1.29576	1.26861	1.28211	1.27958	1.34446	1.21671	1.27899	1.27733	1.28376	1.28275
5	5	1.15686	1.15394	1.15540	1.15718	1.22628	1.06571	1.14318	1.14348	1.15227	1.14281
6	5	1.29885	1.29900	1.29893	1.29611	1.36519	1.18589	1.27239	1.27244	1.29548	1.29399
1	6	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	6	1.39164	1.22166	1.30388	1.29242	1.39164	1.22166	1.30388	1.29242	1.31098	1.30388
3	6	1.22178	1.12981	1.17489	1.17396	1.25257	1.05876	1.15159	1.14046	1.16554	1.15159
4	6	1.44251	1.31595	1.37778	1.37391	1.50699	1.25245	1.37384	1.36073	1.39106	1.37384

5	6	1.36006	1.18481	1.26941	1.26930	1.38245	1.12163	1.24523	1.23252	1.26646	1.25428
6	6	1.33890	1.21385	1.27484	1.27390	1.38772	1.11708	1.24507	1.23232	1.26886	1.25412
1	7	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	7	1.34108	1.18579	1.26105	1.24052	1.34108	1.18579	1.26105	1.24052	1.24998	1.26105
3	7	1.16473	1.05154	1.10669	1.10140	1.21160	1.03449	1.11955	1.09931	1.10632	1.11955
4	7	1.15777	1.08526	1.12093	1.11635	1.25418	1.02377	1.13313	1.11271	1.12257	1.13313
5	7	1.27857	1.21396	1.24585	1.24441	1.40919	1.10546	1.24812	1.22775	1.24618	1.24812
6	7	1.16724	1.06722	1.11611	1.11371	1.29886	0.98785	1.13273	1.11312	1.12442	1.13599
1	8	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	8	1.17083	1.15923	1.16501	1.16556	1.17083	1.15923	1.16501	1.16556	0.91727	1.16501
3	8	1.15211	1.11885	1.13536	1.13510	1.15792	1.12961	1.14367	1.14349	0.99912	1.14367
4	8	1.02823	0.99631	1.01215	1.01276	1.07022	1.01120	1.04029	1.04008	0.99714	1.04029
5	8	1.23369	1.21298	1.22329	1.22287	1.31264	1.15293	1.23020	1.22990	1.07794	1.22329
6	8	1.08462	1.05909	1.07178	1.07241	1.21200	1.00477	1.10353	1.10473	1.12558	1.09604
1	9	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	9	1.14248	1.10310	1.12262	1.12333	1.14248	1.10310	1.12262	1.12333	1.11489	1.12262
3	9	1.24526	1.16240	1.20312	1.20144	1.24737	1.18134	1.21391	1.21481	1.20187	1.21391
4	9	1.24783	1.22435	1.23603	1.23613	1.29599	1.20422	1.24927	1.25078	1.23805	1.24701
5	9	1.33579	1.23154	1.28261	1.28165	1.35658	1.23101	1.29227	1.29321	1.28087	1.29501
6	9	1.31824	1.29386	1.30599	1.30586	1.42107	1.25797	1.33704	1.33840	1.31605	1.30599
1	10	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	10	1.05802	1.01422	1.03589	1.03679	1.05802	1.01422	1.03589	1.03679	1.03745	1.03589
3	10	1.10122	1.06718	1.08407	1.08219	1.14481	1.05881	1.10097	1.10408	1.09587	1.08407
4	10	1.21299	1.20960	1.21129	1.21150	1.23038	1.16216	1.19578	1.20230	1.19357	1.21129
5	10	1.19970	1.09432	1.14580	1.14499	1.20798	1.12170	1.16404	1.17143	1.15553	1.17915
6	10	1.29717	1.26337	1.28016	1.28015	1.32255	1.22509	1.27289	1.28047	1.27244	1.28940
1	11	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	11	1.02552	1.02452	1.02502	1.02479	1.02552	1.02452	1.02502	1.02479	1.03130	1.02502
3	11	1.04068	1.04095	1.04081	1.04058	1.03426	1.04261	1.03842	1.03856	1.04307	1.03842
4	11	1.21879	1.21606	1.21742	1.21737	1.19912	1.23241	1.21565	1.21542	1.21782	1.21742
5	11	1.08609	1.04101	1.06331	1.06158	1.05063	1.05983	1.05522	1.05462	1.05707	1.04532
6	11	1.17818	1.15325	1.16565	1.16445	1.15826	1.16658	1.16241	1.16159	1.16249	1.15150
1	12	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	12	1.07104	1.06784	1.06944	1.06922	1.07104	1.06784	1.06944	1.06922	1.06907	1.06944
3	12	1.04248	1.03372	1.03809	1.03803	1.04200	1.03319	1.03759	1.03744	1.03607	1.03759
4	12	1.16428	1.15713	1.16070	1.16065	1.18173	1.14757	1.16453	1.16408	1.16277	1.16070
5	12	1.04311	1.03999	1.04155	1.04147	1.05685	1.02477	1.04069	1.04050	1.03999	1.03727
6	12	1.21874	1.20829	1.21350	1.21311	1.23554	1.20465	1.22000	1.21948	1.21593	1.21140
Mean		1.13650	1.10010	1.11800	1.11700	1.15600	1.08170	1.11760	1.11540	1.11110	1.11780

2. Year Over Year Monthly Indexes Using Maximum Overlap Bilateral Indexes

The data listed in Tables A.1 to A.20 above were used to compute all of the maximum overlap indexes that are listed in Table A.22 below. However, the imputed prices (in italics) listed in Tables A.1 to A.20 were set equal to 0 when computing the year over year maximum overlap indexes that are listed in Table A.22. Thus the year over year maximum overlap indexes do not use any imputed prices. The indexes listed in Table A.22 are discussed in section 3 of the main text.

Table A.22: Year over Year Alternative Indexes Using Maximum Overlap Price Indexes

y	m	$P_{LFB}^{y,m}$	$P_{PFB}^{y,m}$	$P_{FFB}^{y,m}$	$P_{TFB}^{y,m}$	$P_{LCH}^{y,m}$	$P_{PCH}^{y,m}$	$P_{FCH}^{y,m}$	$P_{TCH}^{y,m}$	$P_{GEKS}^{y,m}$	$P_{S}^{y,m}$
1	1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1	0.99746	0.99881	0.99813	0.99817	0.99746	0.99881	0.99813	0.99817	0.99814	0.99813
3	1	1.03276	1.01894	1.02583	1.02591	1.02762	1.01799	1.02280	1.02261	1.02295	1.02280
4	1	1.01159	1.00992	1.01076	1.01072	1.01586	0.99872	1.00725	1.00700	1.00816	1.01076
5	1	1.12212	1.12896	1.12554	1.12582	1.14808	1.10989	1.12883	1.12854	1.12973	1.13415
6	1	1.07410	1.06543	1.06976	1.06889	1.09958	1.04827	1.07362	1.07252	1.07153	1.06944
1	2	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

2	**2**	**1.14673**	**1.05830**	**1.10163**	**1.09970**	**1.14673**	**1.05830**	**1.10163**	**1.09970**	**1.10937**	**1.10163**
3	**2**	**1.19856**	**1.13544**	**1.16657**	**1.16430**	**1.19530**	**1.10240**	**1.14791**	**1.14597**	**1.15856**	**1.14791**
4	**2**	**1.13489**	**1.06908**	**1.10149**	**1.09983**	**1.13690**	**1.04779**	**1.09144**	**1.08957**	**1.10156**	**1.09144**
5	**2**	**1.35079**	**1.25687**	**1.30298**	**1.30238**	**1.35316**	**1.23472**	**1.29259**	**1.29006**	**1.30486**	**1.29259**
6	**2**	**1.36333**	**1.26804**	**1.31482**	**1.31429**	**1.36508**	**1.23771**	**1.29984**	**1.29727**	**1.31271**	**1.29984**
1	**3**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
2	**3**	**0.92742**	**0.91116**	**0.91925**	**0.91910**	**0.92742**	**0.91116**	**0.91925**	**0.91910**	**0.91669**	**0.91925**
3	**3**	**1.00396**	**0.99578**	**0.99986**	**0.99981**	**1.00995**	**0.98455**	**0.99717**	**0.99686**	**0.99852**	**0.99717**
4	**3**	**1.00033**	**0.99176**	**0.99603**	**0.99611**	**1.00911**	**0.98358**	**0.99626**	**0.99588**	**0.99724**	**0.99626**
5	**3**	**1.09845**	**1.06264**	**1.08040**	**1.07833**	**1.10327**	**1.05318**	**1.07793**	**1.07685**	**1.08213**	**1.09208**
6	**3**	**1.13016**	**1.11723**	**1.12368**	**1.12351**	**1.15473**	**1.09091**	**1.12237**	**1.12130**	**1.12515**	**1.12368**
1	**4**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
2	**4**	**0.98803**	**0.98284**	**0.98543**	**0.98569**	**0.98803**	**0.98284**	**0.98543**	**0.98569**	**0.98766**	**0.98543**
3	**4**	**1.06459**	**1.06038**	**1.06248**	**1.06235**	**1.06796**	**1.04900**	**1.05844**	**1.05817**	**1.06550**	**1.06248**
4	**4**	**1.20496**	**1.18402**	**1.19444**	**1.19482**	**1.19860**	**1.16073**	**1.17951**	**1.17928**	**1.19142**	**1.18402**
5	**4**	**1.22481**	**1.18576**	**1.20513**	**1.20454**	**1.23398**	**1.15532**	**1.19400**	**1.19293**	**1.20392**	**1.20245**
6	**4**	**1.22173**	**1.17182**	**1.19652**	**1.19732**	**1.24896**	**1.13499**	**1.19061**	**1.18951**	**1.19466**	**1.17841**
1	**5**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
2	**5**	**0.95007**	**0.91814**	**0.93397**	**0.93252**	**0.95007**	**0.91814**	**0.93397**	**0.93252**	**0.94462**	**0.93397**
3	**5**	**1.05674**	**1.03102**	**1.04380**	**1.04354**	**1.06935**	**0.99802**	**1.03307**	**1.03104**	**1.05052**	**1.03307**
4	**5**	**1.33870**	**1.26554**	**1.30161**	**1.29967**	**1.33429**	**1.21827**	**1.27496**	**1.27191**	**1.29677**	**1.27496**
5	**5**	**1.17963**	**1.17093**	**1.17527**	**1.17658**	**1.21701**	**1.06707**	**1.13958**	**1.13863**	**1.16610**	**1.13587**
6	**5**	**1.34224**	**1.29900**	**1.32044**	**1.31917**	**1.36461**	**1.18740**	**1.27293**	**1.27122**	**1.31228**	**1.28980**
1	**6**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
2	**6**	**1.39305**	**1.22166**	**1.30455**	**1.29307**	**1.39305**	**1.22166**	**1.30455**	**1.29307**	**1.31228**	**1.30455**
3	**6**	**1.22178**	**1.12981**	**1.17489**	**1.17396**	**1.25384**	**1.05675**	**1.15109**	**1.13948**	**1.16557**	**1.15109**
4	**6**	**1.44355**	**1.31595**	**1.37827**	**1.37444**	**1.50911**	**1.25008**	**1.37350**	**1.35982**	**1.39145**	**1.37350**
5	**6**	**1.36089**	**1.18481**	**1.26980**	**1.26966**	**1.38439**	**1.11951**	**1.24492**	**1.23170**	**1.26652**	**1.25386**
6	**6**	**1.33968**	**1.21385**	**1.27521**	**1.27426**	**1.38967**	**1.11496**	**1.24476**	**1.23149**	**1.26903**	**1.25369**
1	**7**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
2	**7**	**1.35835**	**1.18579**	**1.26914**	**1.24812**	**1.35835**	**1.18579**	**1.26914**	**1.24812**	**1.25497**	**1.26914**
3	**7**	**1.16473**	**1.05154**	**1.10669**	**1.10140**	**1.22720**	**1.01345**	**1.11522**	**1.09393**	**1.10445**	**1.11522**
4	**7**	**1.15635**	**1.08526**	**1.12024**	**1.11432**	**1.27371**	**1.00296**	**1.13025**	**1.10813**	**1.12406**	**1.13025**
5	**7**	**1.28326**	**1.21396**	**1.24813**	**1.24583**	**1.43113**	**1.08299**	**1.24495**	**1.22269**	**1.24902**	**1.24495**
6	**7**	**1.16630**	**1.06722**	**1.11566**	**1.11153**	**1.31909**	**0.96776**	**1.12985**	**1.10854**	**1.12591**	**1.13310**
1	**8**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
2	**8**	**1.17882**	**1.15923**	**1.16899**	**1.16969**	**1.17882**	**1.15923**	**1.16899**	**1.16969**	**1.15923**	**1.16899**
3	**8**	**1.15211**	**1.12012**	**1.13600**	**1.13583**	**1.16583**	**1.12447**	**1.14496**	**1.14526**	**1.13180**	**1.14496**
4	**8**	**1.02645**	**0.99631**	**1.01127**	**1.01139**	**1.07223**	**1.00660**	**1.03889**	**1.03874**	**1.02300**	**1.03889**
5	**8**	**1.24152**	**1.21298**	**1.22717**	**1.22682**	**1.31511**	**1.14769**	**1.22855**	**1.22832**	**1.21689**	**1.23551**
6	**8**	**1.08548**	**1.05909**	**1.07220**	**1.07226**	**1.21428**	**1.00020**	**1.10205**	**1.10331**	**1.08186**	**1.09457**
1	**9**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
2	**9**	**1.14248**	**1.10310**	**1.12262**	**1.12333**	**1.14248**	**1.10310**	**1.12262**	**1.12333**	**1.11489**	**1.12262**
3	**9**	**1.24526**	**1.16240**	**1.20312**	**1.20144**	**1.24737**	**1.18134**	**1.21391**	**1.21481**	**1.20187**	**1.21391**
4	**9**	**1.24783**	**1.22435**	**1.23603**	**1.23613**	**1.29599**	**1.20422**	**1.24927**	**1.25078**	**1.23805**	**1.24701**
5	**9**	**1.33579**	**1.23154**	**1.28261**	**1.28165**	**1.35658**	**1.23101**	**1.29227**	**1.29321**	**1.28087**	**1.29501**
6	**9**	**1.31824**	**1.29386**	**1.30599**	**1.30586**	**1.42107**	**1.25797**	**1.33704**	**1.33840**	**1.31605**	**1.30599**
1	**10**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
2	**10**	**1.08104**	**1.01422**	**1.04710**	**1.04526**	**1.08104**	**1.01422**	**1.04710**	**1.04526**	**1.04292**	**1.04710**
3	**10**	**1.14138**	**1.06718**	**1.10366**	**1.09872**	**1.16972**	**1.05881**	**1.11288**	**1.11310**	**1.10523**	**1.11288**
4	**10**	**1.21299**	**1.20960**	**1.21129**	**1.21150**	**1.25714**	**1.09468**	**1.17310**	**1.17560**	**1.18322**	**1.21129**
5	**10**	**1.11588**	**1.09432**	**1.10505**	**1.10521**	**1.21812**	**1.05657**	**1.13448**	**1.13676**	**1.14941**	**1.17489**
6	**10**	**1.29717**	**1.26337**	**1.28016**	**1.28015**	**1.33365**	**1.16062**	**1.24413**	**1.24654**	**1.26320**	**1.29090**
1	**11**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
2	**11**	**1.02936**	**1.02452**	**1.02694**	**1.02660**	**1.02936**	**1.02452**	**1.02694**	**1.02660**	**1.03219**	**1.02694**
3	**11**	**1.04420**	**1.04095**	**1.04257**	**1.04220**	**1.03812**	**1.03314**	**1.03563**	**1.03532**	**1.04139**	**1.03563**
4	**11**	**1.22044**	**1.21606**	**1.21825**	**1.21813**	**1.21418**	**1.21225**	**1.21322**	**1.21259**	**1.21672**	**1.21825**
5	**11**	**1.05775**	**1.04101**	**1.04934**	**1.04823**	**1.05385**	**1.04249**	**1.04816**	**1.04730**	**1.05206**	**1.04727**
6	**11**	**1.17818**	**1.15325**	**1.16565**	**1.16445**	**1.16180**	**1.15211**	**1.15694**	**1.15584**	**1.15950**	**1.16565**
1	**12**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
2	**12**	**1.07104**	**1.06784**	**1.06944**	**1.06922**	**1.07104**	**1.06784**	**1.06944**	**1.06922**	**1.06907**	**1.06944**
3	**12**	**1.04248**	**1.03372**	**1.03809**	**1.03803**	**1.04200**	**1.03319**	**1.03759**	**1.03744**	**1.03607**	**1.03759**
4	**12**	**1.16428**	**1.15713**	**1.16070**	**1.16065**	**1.18173**	**1.14757**	**1.16453**	**1.16408**	**1.16277**	**1.16070**
5	**12**	**1.04311**	**1.03999**	**1.04155**	**1.04147**	**1.05685**	**1.02477**	**1.04069**	**1.04050**	**1.03999**	**1.03727**
6	**12**	**1.21874**	**1.20829**	**1.21350**	**1.21311**	**1.23554**	**1.20465**	**1.22000**	**1.21948**	**1.21593**	**1.21140**
Mean		**1.13810**	**1.10030**	**1.11890**	**1.11770**	**1.15910**	**1.07650**	**1.11630**	**1.11360**	**1.11870**	**1.11840**

In order to fit all 10 maximum overlap indexes in one row, the index titles have omitted the asterisk; i.e., $P_{LFB}{}^{y,m}$ in row 1 of the above Table should be listed as $P_{LFB}{}^{y,m*}$, $P_{PFB}{}^{y,m}$ should be listed as $P_{PFB}{}^{y,m*}$ and so on.

Our best indexes are the fixed base Fisher and Törnqvist Theil indexes, $P_{FFB}{}^{y,m*}$ and $P_{TFT}{}^{y,m*}$, the GEKS indexes, $P_{GEKS}{}^{y,m*}$ and the Predicted Share Price Similarity linked indexes, $P_{S}{}^{y,m*}$. The average index value over all observations for these four maximum overlap indexes is 1.1184. The average index value for the corresponding four carry forward indexes is 1.1160. Thus the use of carry forward prices led to a downward bias for our best indexes of about 0.24 percentage points per observation.

3. Listing of the Data Using Month to Month Carry Forward and Backward Prices

Tables A23 and A24 below list the price and quantity data for fresh fruit purchased by households in Israel for the 72 months in the years 2012-2017. Carry forward (and backward) unit value prices are used for the prices which were not sold in month t. Note that these new carry forward prices are different from the *year over year carry forward prices* which were listed above in various Tables. The new carry forward prices are *month to month carry forward prices*. In Table A23 below, these carry forward prices are listed in italics. For example if a product n is present in month t but then is missing for the subsequent 3 months, then the last existing price $p_{t,n}$ is carried forward for the next 4 months; i.e., we have $p_{t+1,n} \equiv p_{t+2,n} \equiv p_{t+3,n} \equiv p_{t,n}$. There are 451 carry forward prices listed in Table A23. The maximum number of monthly product prices is 1008 = 72×14. Thus the sample probability that a price listed in Table A23 is an imputed price is 0.447 = 451/1008. The earlier year over year carry forward/backward prices do not coincide with the month to month carry forward/backward prices listed below in italics.

Table A23: Month to Month Price Data using Carry Forward and Backward Prices

t	$p_{t,1}$	$p_{t,2}$	$p_{t,3}$	$p_{t,4}$	$p_{t,5}$	$p_{t,6}$	$p_{t,7}$	$p_{t,8}$	$p_{t,9}$	$p_{t,10}$	$p_{t,11}$	$p_{t,12}$	$p_{t,13}$	$p_{t,14}$
1	5.41	8.29	*4.17*	9.46	4.88	6.22	*9.16*	11.30	*23.78*	*11.92*	*11.43*	5.81	11.82	*10.18*
2	4.99	8.37	*4.17*	10.20	4.84	6.90	*9.16*	15.08	*23.78*	*11.92*	*11.43*	6.32	12.78	*10.18*
3	5.14	8.59	*4.17*	10.76	4.92	7.42	*9.16*	18.67	*23.78*	*11.92*	*11.43*	6.62	13.34	*10.18*
4	5.08	9.06	4.17	*10.76*	5.13	7.25	9.16	18.24	*23.78*	11.92	*11.43*	7.01	13.70	*10.18*
5	5.19	11.48	4.14	*10.76*	5.27	7.05	11.50	16.68	23.78	12.16	11.43	*7.01*	13.69	*10.18*
6	5.66	11.83	3.24	*10.76*	5.57	5.92	10.08	*16.68*	17.44	8.82	11.05	*7.01*	13.74	10.18
7	7.40	13.02	3.18	*10.76*	*5.57*	6.65	9.27	*16.68*	20.10	*8.82*	8.82	*7.01*	*13.74*	10.41
8	10.62	18.23	3.28	*10.76*	*5.57*	8.24	9.06	*16.68*	*20.10*	*8.82*	8.13	*7.01*	*13.74*	9.15
9	9.27	11.65	*3.28*	*10.76*	*5.57*	8.55	8.03	*16.68*	*20.10*	*8.82*	8.15	6.88	*13.74*	10.03
10	8.15	11.26	*3.28*	11.45	6.59	7.93	9.18	*16.68*	*20.10*	*8.82*	8.18	6.19	14.63	9.87
11	7.30	8.89	*3.28*	9.80	5.96	6.09	*9.18*	*16.68*	*20.10*	*8.82*	8.78	5.96	10.45	10.32
12	6.78	8.34	*3.28*	9.48	5.34	5.63	*9.18*	*16.68*	*20.10*	*8.82*	*8.78*	5.63	10.27	*10.32*
13	6.28	8.70	*3.28*	10.55	5.21	5.57	*9.18*	*16.68*	*20.10*	*8.82*	*8.78*	5.89	10.72	*10.32*
14	5.93	9.15	*3.28*	11.41	*5.21*	*5.57*	*9.18*	23.29	*20.10*	*8.82*	*8.78*	6.43	11.58	*10.32*
15	5.70	9.43	*3.28*	11.69	5.16	6.11	*9.18*	15.31	*20.10*	*8.82*	*8.78*	6.64	11.72	*10.32*
16	6.84	11.00	*3.28*	*11.69*	5.43	6.26	*9.18*	16.62	*20.10*	*8.82*	*8.78*	7.18	12.42	*10.32*
17	7.35	14.62	3.49	*11.69*	5.67	5.96	11.08	*16.62*	*20.10*	9.46	*8.78*	*7.18*	*12.42*	*10.32*
18	7.83	19.58	3.36	*11.69*	5.86	5.85	11.25	*16.62*	42.05	14.44	12.61	*7.18*	*12.42*	*10.32*
19	9.96	*19.58*	3.30	*11.69*	*5.86*	7.46	11.91	*16.62*	53.77	*14.44*	10.65	*7.18*	*12.42*	10.92
20	9.44	*19.58*	3.83	*11.69*	*5.86*	7.78	11.53	*16.62*	*53.77*	*14.44*	10.94	*7.18*	*12.42*	10.35
21	8.00	11.92	*3.83*	*11.69*	*5.86*	7.21	10.07	*16.62*	*53.77*	*14.44*	11.31	7.36	*12.42*	10.85
22	8.03	9.80	*3.83*	12.30	6.43	6.80	*10.07*	*16.62*	*53.77*	*14.44*	12.32	7.19	15.47	13.09
23	7.46	8.90	*3.83*	9.82	5.89	6.07	*10.07*	*16.62*	*53.77*	*14.44*	*12.32*	6.43	14.12	*13.09*
24	6.92	8.61	*3.83*	10.19	5.28	5.79	*10.07*	*16.62*	*53.77*	*14.44*	*12.32*	6.46	14.64	*13.09*
25	6.63	8.88	*3.83*	10.49	5.03	5.44	*10.07*	*16.62*	*53.77*	*14.44*	*12.32*	6.30	14.94	*13.09*
26	5.97	8.84	*3.83*	11.32	*5.03*	5.98	*10.07*	25.11	*53.77*	*14.44*	*12.32*	6.50	14.92	*13.09*
27	5.72	9.47	*3.83*	12.41	4.97	6.51	*10.07*	18.23	*53.77*	*14.44*	*12.32*	6.82	15.36	*13.09*

28	6.00	10.27	*3.83*	*12.41*	5.09	7.60	*10.07*	17.80	*53.77*	*14.44*	*12.32*	7.72	16.91	*13.09*
29	6.60	13.66	4.10	*12.41*	5.34	*7.60*	10.62	*17.80*	40.84	14.79	*12.32*	*7.72*	19.93	*13.09*
30	6.64	14.00	2.55	*12.41*	5.63	8.07	10.42	*17.80*	32.81	13.25	14.03	*7.72*	27.25	*13.09*
31	7.51	15.44	2.29	*12.41*	*5.63*	10.76	11.23	*17.80*	37.98	*13.25*	12.59	*7.72*	*27.25*	10.63
32	8.23	19.44	3.12	*12.41*	*5.63*	10.56	11.84	*17.80*	22.50	*13.25*	13.30	7.72	*27.25*	8.94
33	7.12	12.03	*3.12*	*12.41*	*5.63*	9.34	11.43	*17.80*	*22.50*	*13.25*	13.44	7.20	*27.25*	9.66
34	7.14	10.51	*3.12*	12.98	6.21	8.79	*11.43*	*17.80*	*22.50*	*13.25*	14.50	7.33	17.43	9.67
35	6.73	8.58	*3.12*	9.80	5.78	6.40	*11.43*	*17.80*	*22.50*	*13.25*	*14.50*	6.62	14.98	12.26
36	6.47	7.80	*3.12*	8.98	5.44	6.09	*11.43*	*17.80*	*22.50*	*13.25*	*14.50*	6.26	14.61	*12.26*
37	6.20	8.00	*3.12*	8.94	4.99	6.27	*11.43*	*17.80*	*22.50*	*13.25*	*14.50*	5.83	14.98	*12.26*
38	5.97	8.15	*3.12*	9.95	5.14	6.06	*11.43*	23.49	*22.50*	*13.25*	*14.50*	5.94	15.41	*12.26*
39	6.08	9.06	*3.12*	11.02	4.98	6.83	*11.43*	18.95	*22.50*	*13.25*	*14.50*	6.17	15.73	*12.26*
40	7.04	12.60	*3.12*	*11.02*	5.41	9.68	*11.43*	18.35	*22.50*	*13.25*	*14.50*	7.03	16.30	*12.26*
41	7.73	15.92	4.56	*11.02*	5.39	13.19	11.75	*18.35*	61.43	17.78	*14.50*	*7.03*	17.16	*12.26*
42	8.62	18.98	3.68	*11.02*	*5.39*	12.71	10.80	*18.35*	34.48	12.36	13.56	*7.03*	21.55	*12.26*
43	9.83	*18.98*	2.51	*11.02*	*5.39*	15.91	10.24	*18.35*	30.62	*12.36*	10.36	*7.03*	*21.55*	12.32
44	9.87	*18.98*	*2.51*	*11.02*	*5.39*	12.25	10.14	*18.35*	*30.62*	*12.36*	9.61	*7.03*	*21.55*	10.40
45	9.42	12.23	*2.51*	*11.02*	*5.39*	9.59	10.85	*18.35*	*30.62*	*12.36*	11.18	7.89	*21.55*	12.29
46	9.53	11.54	*2.51*	12.85	7.28	8.91	11.94	*18.35*	*30.62*	*12.36*	11.58	7.22	20.19	11.88
47	8.82	10.13	*2.51*	12.40	6.27	7.75	*11.94*	*18.35*	*30.62*	*12.36*	11.68	6.62	15.13	*11.88*
48	7.87	11.09	*2.51*	11.94	5.52	6.00	*11.94*	*18.35*	*30.62*	*12.36*	*11.68*	6.20	14.36	*11.88*
49	7.07	11.13	*2.51*	12.59	5.35	6.12	*11.94*	*18.35*	*30.62*	*12.36*	*11.68*	5.93	14.59	*11.88*
50	6.99	12.27	*2.51*	13.22	5.09	7.22	*11.94*	26.86	*30.62*	*12.36*	*11.68*	6.15	14.88	*11.88*
51	6.78	13.98	*2.51*	*13.22*	5.13	7.51	*11.94*	18.06	*30.62*	*12.36*	*11.68*	6.03	15.11	*11.88*
52	7.05	18.26	*2.51*	*13.22*	5.07	8.40	*11.94*	18.80	*30.62*	*12.36*	*11.68*	6.58	16.36	*11.88*
53	7.52	19.36	4.07	*13.22*	5.81	8.98	11.27	*18.80*	39.10	18.31	*11.68*	*6.58*	17.33	*11.88*
54	9.01	20.42	2.67	*13.22*	*5.81*	10.99	9.73	*18.80*	34.21	15.05	13.62	*6.58*	22.38	*11.88*
55	11.34	*20.42*	3.09	*13.22*	*5.81*	12.56	10.66	*18.80*	37.31	15.05	12.85	*6.58*	*22.38*	11.35
56	10.30	*20.42*	4.01	*13.22*	*5.81*	9.65	10.73	*18.80*	*37.31*	15.05	13.20	*6.58*	*22.38*	11.19
57	8.91	13.52	*4.01*	*13.22*	*5.81*	8.39	11.77	*18.80*	*37.31*	15.05	14.61	7.40	*22.38*	11.52
58	8.35	12.65	*4.01*	13.61	6.57	7.88	*11.77*	*18.80*	*37.31*	15.05	14.61	7.07	22.85	12.87
59	7.26	9.58	*4.01*	11.31	6.00	5.89	*11.77*	*18.80*	*37.31*	15.05	14.61	6.17	19.60	*12.87*
60	6.70	9.15	*4.01*	10.85	5.45	5.31	*11.77*	*18.80*	*37.31*	15.05	14.61	5.83	17.21	*12.87*
61	6.51	9.64	*4.01*	11.11	5.25	6.07	*11.77*	*18.80*	*37.31*	15.05	14.61	*5.83*	17.88	*12.87*
62	6.39	10.59	*4.01*	11.85	5.00	8.23	*11.77*	28.26	*37.31*	15.05	14.61	5.65	18.97	*12.87*
63	6.32	11.05	*4.01*	13.66	5.24	8.85	*11.77*	19.26	*37.31*	15.05	14.61	6.06	19.66	*12.87*
64	6.47	12.59	*4.01*	*13.66*	5.45	10.75	*11.77*	16.85	*37.31*	15.05	14.61	6.28	20.39	*12.87*
65	7.00	15.34	4.77	*13.66*	6.16	12.30	12.95	*16.85*	*37.31*	18.03	14.61	*6.28*	22.56	*12.87*
66	8.20	18.56	2.93	*13.66*	*6.16*	11.15	9.81	*16.85*	31.08	14.71	14.01	*6.28*	26.03	*12.87*
67	10.86	*18.56*	2.32	*13.66*	*6.16*	14.74	9.87	*16.85*	35.14	*14.71*	11.23	*6.28*	*26.03*	13.48
68	10.87	*18.56*	2.60	*13.66*	*6.16*	12.20	10.39	*16.85*	*35.14*	*14.71*	11.09	*6.28*	*26.03*	11.37
69	10.11	17.74	*2.60*	*13.66*	*6.16*	10.52	10.72	*16.85*	*35.14*	*14.71*	12.03	7.98	*26.03*	12.49
70	9.62	14.86	*2.60*	13.85	6.81	8.92	12.67	*16.85*	*35.14*	*14.71*	13.13	7.09	21.74	*12.49*
71	8.49	11.51	*2.60*	11.06	6.95	6.57	*12.67*	*16.85*	*35.14*	*14.71*	12.93	6.34	17.18	12.26
72	7.38	12.96	*2.60*	10.94	6.35	6.38	*12.67*	*16.85*	*35.14*	*14.71*	*12.93*	6.15	16.26	*12.26*

The monthly quantity data are listed in Table A24 below. The quantity data listed in Table A24 are the same as the quantity data that were listed earlier in various Tables in this Appendix but the earlier data was listed as year over year data for each month. The data listed below are month to month data that start at January 2012 and end at December 2017.

Table A24: Monthly Quantity Data for Household Fresh Fruit Consumption

t	$q_{t,1}$	$q_{t,2}$	$q_{t,3}$	$q_{t,4}$	$q_{t,5}$	$q_{t,6}$	$q_{t,7}$	$q_{t,8}$	$q_{t,9}$	$q_{t,10}$	$q_{t,11}$	$q_{t,12}$	$q_{t,13}$	$q_{t,14}$
1	0.370	0.676	0.000	0.465	0.082	1.881	0.000	0.000	0.000	0.000	0.000	1.824	0.135	0.000
2	0.701	0.920	0.000	0.510	0.103	2.087	0.000	1.134	0.000	0.000	0.000	1.408	0.102	0.000
3	0.661	0.908	0.000	0.362	0.081	1.819	0.000	0.884	0.000	0.000	0.000	1.269	0.112	0.000
4	0.689	0.585	0.000	0.000	0.156	1.876	0.000	0.609	0.000	0.000	0.000	0.728	0.131	0.000
5	0.751	0.409	4.106	0.000	0.076	1.730	0.922	0.456	0.000	0.206	0.000	0.000	0.080	0.000
6	0.724	0.440	6.698	0.000	0.036	1.486	1.657	0.000	0.717	1.270	0.290	0.000	0.022	0.000
7	0.595	0.292	8.145	0.000	0.000	0.722	2.093	0.000	0.488	0.000	0.964	0.000	0.000	0.221
8	0.452	0.159	6.159	0.000	0.000	0.558	1.932	0.000	0.000	0.000	1.009	0.000	0.000	0.721
9	0.647	0.335	0.000	0.000	0.000	0.643	2.379	0.000	0.000	0.000	1.104	0.058	0.000	0.857
10	0.724	0.409	0.000	0.428	0.030	1.021	1.569	0.000	0.000	0.000	0.648	0.420	0.055	0.395

11	0.712	0.765	0.000	0.510	0.101	2.069	0.000	0.000	0.000	0.000	0.410	1.309	0.124	0.223
12	0.678	0.923	0.000	0.390	0.150	2.274	0.000	0.000	0.000	0.000	0.000	1.545	0.127	0.000
13	0.430	0.897	0.000	0.417	0.058	1.957	0.000	0.000	0.000	0.000	0.000	1.579	0.103	0.000
14	0.624	0.831	0.000	0.412	0.269	2.621	0.000	0.593	0.000	0.000	0.000	1.664	0.155	0.000
15	0.684	0.732	0.000	0.257	0.116	2.242	0.000	0.947	0.000	0.000	0.000	1.160	0.154	0.000
16	0.760	0.591	0.000	0.000	0.092	2.141	0.000	0.698	0.000	0.000	0.000	0.766	0.129	0.000
17	0.626	0.321	6.504	0.000	0.053	1.913	1.273	0.000	0.000	0.370	0.000	0.000	0.000	0.000
18	0.766	0.266	7.738	0.000	0.051	1.419	1.831	0.000	0.228	0.616	0.523	0.000	0.000	0.000
19	0.612	0.000	7.394	0.000	0.000	0.871	1.520	0.000	0.073	0.000	0.761	0.000	0.000	0.421
20	0.625	0.000	5.065	0.000	0.000	0.746	1.761	0.000	0.000	0.000	0.914	0.000	0.000	0.850
21	0.650	0.235	0.000	0.000	0.000	0.957	1.927	0.000	0.000	0.000	0.716	0.231	0.000	0.710
22	0.635	0.673	0.000	0.537	0.078	1.721	0.000	0.000	0.000	0.000	0.373	0.612	0.045	0.306
23	0.603	1.000	0.000	0.601	0.085	2.405	0.000	0.000	0.000	0.000	0.000	1.726	0.064	0.000
24	0.621	1.010	0.000	0.481	0.170	2.522	0.000	0.000	0.000	0.000	0.000	1.718	0.089	0.000
25	0.513	0.890	0.000	0.486	0.298	2.261	0.000	0.000	0.000	0.000	0.000	1.175	0.067	0.000
26	0.754	1.075	0.000	0.486	0.119	2.308	0.000	0.737	0.000	0.000	0.000	1.492	0.168	0.000
27	0.822	0.612	0.000	0.290	0.121	2.012	0.000	0.845	0.000	0.000	0.000	1.496	0.085	0.000
28	0.617	0.662	0.000	0.000	0.157	1.737	0.000	0.663	0.000	0.000	0.000	0.997	0.053	0.000
29	0.682	0.417	5.244	0.000	0.075	1.526	1.525	0.000	0.088	0.176	0.000	0.000	0.045	0.000
30	0.678	0.450	8.118	0.000	0.036	1.016	1.910	0.000	0.466	0.694	0.335	0.000	0.011	0.000
31	0.746	0.389	9.869	0.000	0.000	0.539	1.915	0.000	0.179	0.000	0.667	0.000	0.000	0.546
32	0.656	0.180	5.577	0.000	0.000	0.616	1.791	0.000	0.031	0.000	0.759	0.000	0.000	1.040
33	0.758	0.532	0.000	0.000	0.000	0.921	1.899	0.000	0.000	0.000	0.543	0.181	0.000	0.932
34	0.742	0.666	0.000	0.108	0.048	1.365	0.000	0.000	0.000	0.000	0.269	0.641	0.040	0.486
35	0.594	0.897	0.000	0.510	0.087	2.328	0.000	0.000	0.000	0.000	0.000	1.344	0.080	0.220
36	0.649	1.077	0.000	0.657	0.092	2.463	0.000	0.000	0.000	0.000	0.000	1.534	0.110	0.000
37	0.645	0.975	0.000	0.559	0.160	2.281	0.000	0.000	0.000	0.000	0.000	1.492	0.100	0.000
38	0.553	1.031	0.000	0.412	0.156	2.591	0.000	0.766	0.000	0.000	0.000	1.919	0.117	0.000
39	0.658	0.828	0.000	0.209	0.181	2.255	0.000	0.813	0.000	0.000	0.000	1.556	0.108	0.000
40	0.895	0.683	0.000	0.000	0.129	1.550	0.000	0.687	0.000	0.000	0.000	1.252	0.135	0.000
41	0.660	0.528	4.211	0.000	0.056	1.054	1.183	0.000	0.016	0.107	0.000	0.000	0.041	0.000
42	0.673	0.295	7.038	0.000	0.000	0.653	1.880	0.000	0.299	0.777	0.354	0.000	0.014	0.000
43	0.600	0.000	8.486	0.000	0.000	0.289	2.129	0.000	0.349	0.000	0.685	0.000	0.000	0.390
44	0.719	0.000	7.371	0.000	0.000	0.498	1.765	0.000	0.000	0.000	0.832	0.000	0.000	0.673
45	0.594	0.278	0.000	0.000	0.000	0.792	1.604	0.000	0.000	0.000	0.689	0.114	0.000	0.667
46	0.724	0.537	0.000	0.117	0.041	1.459	1.508	0.000	0.000	0.000	0.717	0.402	0.064	0.438
47	0.612	1.066	0.000	0.435	0.080	2.400	0.000	0.000	0.000	0.000	0.283	1.148	0.099	0.000
48	0.635	0.938	0.000	0.553	0.163	2.983	0.000	0.000	0.000	0.000	0.000	1.645	0.104	0.000
49	0.552	1.006	0.000	0.485	0.093	2.647	0.000	0.000	0.000	0.000	0.000	1.737	0.206	0.000
50	0.658	0.717	0.000	0.386	0.157	2.299	0.000	0.648	0.000	0.000	0.000	1.886	0.108	0.000
51	0.708	0.694	0.000	0.000	0.234	2.490	0.000	1.107	0.000	0.000	0.000	1.509	0.152	0.000
52	0.766	0.460	0.000	0.000	0.079	1.988	0.000	0.585	0.000	0.000	0.000	1.231	0.122	0.000
53	0.785	0.584	5.430	0.000	0.103	1.726	1.287	0.000	0.138	0.284	0.000	0.000	0.069	0.000
54	0.599	0.318	8.876	0.000	0.000	0.792	1.922	0.000	0.406	0.472	0.382	0.000	0.031	0.000
55	0.635	0.000	8.188	0.000	0.000	0.701	2.073	0.000	0.198	0.000	0.545	0.000	0.000	0.643
56	0.718	0.000	4.963	0.000	0.000	0.974	2.171	0.000	0.000	0.000	0.750	0.000	0.000	1.028
57	0.831	0.473	0.000	0.000	0.000	1.335	2.022	0.000	0.000	0.000	0.568	0.243	0.000	0.972
58	1.018	0.735	0.000	0.272	0.046	2.183	0.000	0.000	0.000	0.000	0.000	0.863	0.101	0.420
59	0.992	1.075	0.000	0.557	0.150	2.920	0.000	0.000	0.000	0.000	0.000	1.313	0.087	0.000
60	0.896	1.191	0.000	0.562	0.128	2.976	0.000	0.000	0.000	0.000	0.000	1.955	0.134	0.000
61	0.906	1.172	0.000	0.630	0.057	3.262	0.000	0.000	0.000	0.000	0.000	2.093	0.056	0.000
62	0.657	0.859	0.000	0.447	0.260	2.211	0.000	0.711	0.000	0.000	0.000	2.071	0.111	0.000
63	0.759	0.787	0.000	0.293	0.095	2.395	0.000	1.038	0.000	0.000	0.000	1.650	0.102	0.000
64	0.773	0.627	0.000	0.000	0.037	2.047	0.000	0.926	0.000	0.000	0.000	1.210	0.069	0.000
65	0.814	0.587	5.891	0.000	0.065	1.504	1.243	0.000	0.000	0.322	0.000	0.000	0.040	0.000
66	0.915	0.436	9.693	0.000	0.000	0.969	2.487	0.000	0.560	0.727	0.378	0.000	0.019	0.000
67	0.847	0.000	12.845	0.000	0.000	0.468	2.837	0.000	0.361	0.000	0.784	0.000	0.000	0.593
68	0.690	0.000	8.423	0.000	0.000	0.770	2.348	0.000	0.000	0.000	0.748	0.000	0.000	0.730
69	0.752	0.282	0.000	0.000	0.000	1.502	2.136	0.000	0.000	0.000	0.948	0.188	0.000	0.945
70	0.811	0.505	0.000	0.159	0.044	1.996	1.294	0.000	0.000	0.000	0.457	0.367	0.055	0.456
71	0.836	0.990	0.000	0.443	0.158	3.014	0.000	0.000	0.000	0.000	0.116	1.167	0.076	0.179
72	0.623	0.980	0.000	0.558	0.079	3.119	0.000	0.000	0.000	0.000	0.000	1.447	0.062	0.000

4. Month to Month Fixed Base Fisher Indexes Using Carry Forward Prices

Below is a listing of the 12 Fisher fixed base "star" indexes, $P_{F1}^t – P_{F12}^t$ that were plotted on Chart 5 in section 6 of the main text.

Table A25: Fisher Star Month to Month Indexes using Carry Forward Prices and Using Months 1 to 12 as the Base Month

t	P_{FS1}^t	P_{FS2}^t	P_{FS3}^t	P_{FS4}^t	P_{FS5}^t	P_{FS6}^t	P_{FS7}^t	P_{FS8}^t	P_{FS9}^t	P_{FS10}^t	P_{FS11}^t	P_{FS12}^t
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.0994	1.0994	1.0686	1.0740	1.0586	1.0295	1.0307	1.0398	1.0378	1.0389	1.0305	1.0258
3	1.1794	1.2133	1.1794	1.1829	1.1357	1.0827	1.0795	1.0826	1.0892	1.0996	1.1056	1.1056
4	1.1858	1.2138	1.1823	1.1858	1.1383	1.0945	1.0828	1.0747	1.0751	1.0912	1.1095	1.1148
5	1.1614	1.2061	1.2060	1.2100	1.1614	1.1747	1.1915	1.2146	1.2311	1.2136	1.1656	1.1638
6	0.9984	1.0662	1.0875	1.0817	0.9871	0.9984	1.0361	1.1012	1.1429	1.1232	1.0675	1.0575
7	1.0436	1.1131	1.1402	1.1429	1.0173	1.0057	1.0436	1.1104	1.1525	1.1456	1.1060	1.1075
8	1.1542	1.2204	1.2574	1.2736	1.1037	1.0464	1.0848	1.1542	1.2020	1.2199	1.2130	1.2342
9	1.1276	1.1945	1.2209	1.2436	1.0637	0.9850	1.0210	1.0828	1.1276	1.1383	1.1383	1.1494
10	1.1555	1.2228	1.2393	1.2557	1.1058	1.0271	1.0526	1.0933	1.1446	1.1555	1.1541	1.1687
11	1.0253	1.0939	1.0938	1.0958	1.0216	0.9589	0.9675	0.9757	1.0157	1.0266	1.0253	1.0363
12	0.9759	1.0459	1.0410	1.0381	0.9739	0.9213	0.9196	0.9126	0.9574	0.9649	0.9656	0.9759
13	0.9981	1.0607	1.0553	1.0439	0.9753	0.9233	0.9255	0.9222	0.9693	0.9819	0.9866	0.9968
14	1.1035	1.2111	1.1819	1.1661	1.0744	0.9876	0.9939	1.0049	1.0371	1.0521	1.0519	1.0603
15	1.1110	1.1144	1.0841	1.0780	1.0200	0.9650	0.9694	0.9822	1.0133	1.0319	1.0337	1.0413
16	1.1884	1.2015	1.1723	1.1686	1.0841	1.0162	1.0107	1.0108	1.0508	1.0820	1.1038	1.1211
17	1.1062	1.1716	1.1934	1.1862	1.0876	1.0912	1.1183	1.1525	1.1962	1.1935	1.1554	1.1669
18	1.2036	1.2771	1.3117	1.2985	1.1773	1.3016	1.2999	1.2960	1.3713	1.3492	1.2898	1.2898
19	1.2232	1.2921	1.3299	1.3354	1.2047	1.3339	1.3279	1.3115	1.3828	1.3705	1.3057	1.3150
20	1.2506	1.3192	1.3577	1.3642	1.2387	1.3464	1.3530	1.3466	1.3852	1.3741	1.3144	1.3245
21	1.1918	1.2583	1.2811	1.2863	1.1627	1.2606	1.2597	1.2311	1.2658	1.2496	1.2037	1.1994
22	1.2069	1.2556	1.2586	1.2604	1.1877	1.2721	1.2324	1.1572	1.2070	1.2124	1.2132	1.2095
23	1.0702	1.1240	1.1210	1.1220	1.0739	1.1651	1.1205	1.0407	1.0907	1.0821	1.0742	1.0691
24	1.0486	1.1022	1.0984	1.0942	1.0525	1.1485	1.1047	1.0244	1.0710	1.0619	1.0511	1.0437
25	1.0258	1.0823	1.0785	1.0710	1.0341	1.1335	1.0906	1.0095	1.0547	1.0407	1.0281	1.0192
26	1.1607	1.2825	1.2483	1.2370	1.1669	1.2249	1.1849	1.1131	1.1564	1.1450	1.1198	1.1039
27	1.1834	1.2119	1.1792	1.1720	1.1326	1.2156	1.1748	1.1125	1.1410	1.1424	1.1199	1.1077
28	1.2879	1.2923	1.2602	1.2599	1.1886	1.2635	1.2126	1.1358	1.1821	1.1969	1.2050	1.2054
29	1.2527	1.3145	1.3386	1.3413	1.2263	1.3226	1.3109	1.2998	1.3404	1.3249	1.3039	1.2981
30	1.1914	1.2467	1.2775	1.2873	1.1084	1.1730	1.1704	1.2131	1.3307	1.3118	1.2760	1.2595
31	1.2270	1.2729	1.3080	1.3351	1.1313	1.1693	1.1549	1.1777	1.3088	1.3146	1.2936	1.2985
32	1.3373	1.3868	1.4285	1.4539	1.2558	1.2100	1.2288	1.3112	1.4039	1.4136	1.3891	1.4024
33	1.3217	1.3806	1.4072	1.4316	1.2385	1.1792	1.1932	1.2494	1.3566	1.3520	1.3176	1.3164
34	1.3031	1.3547	1.3609	1.3698	1.2405	1.1753	1.1570	1.1606	1.2651	1.2916	1.2909	1.2963
35	1.0860	1.1488	1.1453	1.1479	1.0744	1.0477	1.0414	1.0547	1.1437	1.1244	1.0935	1.0803
36	1.0179	1.0858	1.0807	1.0837	1.0219	1.0001	0.9960	1.0042	1.0940	1.0667	1.0298	1.0131
37	1.0108	1.0824	1.0775	1.0817	1.0232	0.9999	0.9950	1.0036	1.0919	1.0638	1.0241	1.0070
38	1.1011	1.2110	1.1793	1.1773	1.1071	1.0566	1.0616	1.0908	1.1727	1.1433	1.0810	1.0549
39	1.1584	1.2051	1.1747	1.1773	1.1152	1.0752	1.0748	1.1029	1.1857	1.1681	1.1192	1.0991
40	1.4119	1.4242	1.3954	1.4164	1.2931	1.2099	1.1888	1.2082	1.3192	1.3293	1.3325	1.3388
41	1.5217	1.5808	1.6135	1.6634	1.5004	1.6363	1.5801	1.4940	1.5683	1.5619	1.5812	1.5946
42	1.4085	1.4684	1.5112	1.5556	1.3528	1.3755	1.3805	1.4225	1.5084	1.5057	1.5100	1.5202
43	1.3324	1.3779	1.4209	1.4818	1.2341	1.2007	1.1764	1.2418	1.3607	1.3836	1.4126	1.4374
44	1.2412	1.2990	1.3383	1.3789	1.1610	1.1336	1.1205	1.1676	1.2692	1.2942	1.3039	1.3284
45	1.3495	1.4161	1.4453	1.4780	1.2440	1.2051	1.2102	1.2581	1.3937	1.3852	1.3581	1.3636
46	1.3541	1.4229	1.4419	1.4659	1.2622	1.2282	1.2289	1.2533	1.4023	1.3996	1.3668	1.3716
47	1.2434	1.3065	1.3047	1.3119	1.1768	1.1501	1.1204	1.0958	1.2383	1.2498	1.2473	1.2578
48	1.1275	1.1975	1.1929	1.1847	1.0840	1.0803	1.0526	1.0335	1.1508	1.1505	1.1282	1.1327
49	1.1255	1.1958	1.1908	1.1775	1.0777	1.0724	1.0470	1.0299	1.1509	1.1510	1.1289	1.1309
50	1.3027	1.4325	1.3976	1.3865	1.2447	1.1800	1.1547	1.1564	1.2761	1.2846	1.2533	1.2569
51	1.2963	1.3143	1.2795	1.2781	1.1754	1.1492	1.1225	1.1245	1.2395	1.2492	1.2216	1.2270
52	1.4336	1.4546	1.4304	1.4291	1.2811	1.2347	1.1919	1.1807	1.3107	1.3415	1.3552	1.3816
53	1.3939	1.4744	1.5047	1.5119	1.3743	1.4592	1.4031	1.3792	1.4460	1.4524	1.4553	1.4715
54	1.3128	1.3810	1.4206	1.4458	1.2260	1.2674	1.2374	1.2722	1.3966	1.4066	1.4161	1.4203
55	1.3754	1.4421	1.4850	1.5273	1.3113	1.3296	1.3095	1.3560	1.4524	1.4687	1.4658	1.4771
56	1.3604	1.4399	1.4804	1.5033	1.3337	1.3520	1.3633	1.4133	1.4534	1.4554	1.4279	1.4323

57	**1.3674**	**1.4441**	**1.4697**	**1.4841**	**1.3305**	**1.3415**	**1.3531**	**1.3752**	**1.4462**	**1.4306**	**1.3798**	**1.3695**
58	**1.3659**	**1.4252**	**1.4282**	**1.4344**	**1.3388**	**1.3374**	**1.2922**	**1.2508**	**1.3227**	**1.3423**	**1.3506**	**1.3539**
59	**1.0984**	**1.1747**	**1.1700**	**1.1685**	**1.1289**	**1.1577**	**1.1302**	**1.0881**	**1.1491**	**1.1352**	**1.1050**	**1.0908**
60	**1.0215**	**1.1044**	**1.0988**	**1.0902**	**1.0683**	**1.1033**	**1.0846**	**1.0515**	**1.1074**	**1.0856**	**1.0375**	**1.0165**
61	**1.0698**	**1.1460**	**1.1403**	**1.1359**	**1.1051**	**1.1363**	**1.1139**	**1.0799**	**1.1354**	**1.1203**	**1.0824**	**1.0664**
62	**1.3011**	**1.4455**	**1.4091**	**1.4129**	**1.3279**	**1.2879**	**1.2693**	**1.2612**	**1.3163**	**1.3049**	**1.2671**	**1.2536**
63	**1.3398**	**1.3621**	**1.3252**	**1.3362**	**1.2844**	**1.2801**	**1.2573**	**1.2481**	**1.2996**	**1.3036**	**1.2782**	**1.2665**
64	**1.4533**	**1.4290**	**1.3916**	**1.4189**	**1.3491**	**1.3361**	**1.2993**	**1.2814**	**1.3419**	**1.3635**	**1.3669**	**1.3685**
65	**1.5031**	**1.5489**	**1.5797**	**1.6212**	**1.5094**	**1.5759**	**1.5487**	**1.5459**	**1.6052**	**1.5992**	**1.5819**	**1.5841**
66	**1.3204**	**1.3707**	**1.4099**	**1.4400**	**1.2365**	**1.2729**	**1.2508**	**1.2990**	**1.4122**	**1.4173**	**1.4203**	**1.4176**
67	**1.3047**	**1.3438**	**1.3857**	**1.4406**	**1.2050**	**1.2099**	**1.1705**	**1.2256**	**1.3537**	**1.3791**	**1.3968**	**1.4088**
68	**1.2891**	**1.3368**	**1.3767**	**1.4192**	**1.2056**	**1.2133**	**1.1907**	**1.2359**	**1.3444**	**1.3666**	**1.3650**	**1.3777**
69	**1.4920**	**1.5285**	**1.5649**	**1.5956**	**1.3356**	**1.3166**	**1.2972**	**1.3333**	**1.4726**	**1.4918**	**1.5033**	**1.5197**
70	**1.4483**	**1.4955**	**1.5161**	**1.5364**	**1.3406**	**1.3442**	**1.3178**	**1.3213**	**1.4781**	**1.4792**	**1.4572**	**1.4645**
71	**1.1903**	**1.2457**	**1.2470**	**1.2544**	**1.1544**	**1.1777**	**1.1321**	**1.0989**	**1.2299**	**1.2237**	**1.1952**	**1.1989**
72	**1.1733**	**1.2241**	**1.2237**	**1.2232**	**1.1276**	**1.1575**	**1.1127**	**1.0793**	**1.2075**	**1.1994**	**1.1777**	**1.1843**
Mean	**1.2195**	**1.2762**	**1.2819**	**1.2920**	**1.1743**	**1.1791**	**1.1664**	**1.1698**	**1.2445**	**1.2454**	**1.2290**	**1.2309**

For the final month, the lowest index value is 1.0793 and the highest is 1.2241. The lowest average value for the 12 indexes is 1.1664 and the highest is 1.2920. Thus the choice of a base month matters a lot for our particular empirical example.

5. Maximum Overlap Month to Month Fixed Base Fisher Indexes

Below is a listing of the 12 Fisher fixed base "star" indexes, P_{F1}^{t*} – P_{F12}^{t*} that were plotted on Chart 7 in section 7 of the main text.

Table A26: Fisher Star Maximum Overlap Month to Month Indexes Using Months 1 to 12 as the Base Month

t	P_{FS1}^{t*}	P_{FS2}^{t*}	P_{FS3}^{t*}	P_{FS4}^{t*}	P_{FS5}^{t*}	P_{FS6}^{t*}	P_{FS7}^{t*}	P_{FS8}^{t*}	P_{FS9}^{t*}	P_{FS10}^{t*}	P_{FS11}^{t*}	P_{FS12}^{t*}
1	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
2	**1.0660**	**1.0660**	**1.0154**	**1.0274**	**1.0519**	**1.0409**	**1.0266**	**1.0223**	**1.0253**	**1.0415**	**1.0529**	**1.0541**
3	**1.1206**	**1.1765**	**1.1206**	**1.1340**	**1.1637**	**1.0901**	**1.0730**	**1.0677**	**1.0742**	**1.0922**	**1.1058**	**1.1070**
4	**1.1370**	**1.1798**	**1.1235**	**1.1370**	**1.1589**	**1.1102**	**1.0919**	**1.0887**	**1.0728**	**1.0795**	**1.1144**	**1.1224**
5	**1.1624**	**1.1780**	**1.1192**	**1.1404**	**1.1624**	**1.2880**	**1.4462**	**1.7124**	**1.4438**	**1.2416**	**1.1292**	**1.1443**
6	**1.0919**	**1.1183**	**1.1224**	**1.1182**	**0.9854**	**1.0919**	**1.2534**	**1.5766**	**1.4340**	**1.2197**	**1.0899**	**1.0749**
7	**1.2646**	**1.3132**	**1.3207**	**1.3169**	**1.0164**	**1.1017**	**1.2646**	**1.6030**	**1.4584**	**1.2577**	**1.1568**	**1.2275**
8	**1.6720**	**1.7436**	**1.7548**	**1.7462**	**1.1350**	**1.1580**	**1.3191**	**1.6720**	**1.5190**	**1.3514**	**1.3164**	**1.6277**
9	**1.4099**	**1.4660**	**1.4707**	**1.4942**	**1.1351**	**1.0735**	**1.2226**	**1.5520**	**1.4099**	**1.2481**	**1.2167**	**1.3729**
10	**1.2621**	**1.2918**	**1.2949**	**1.3293**	**1.1815**	**1.1298**	**1.2691**	**1.5616**	**1.4256**	**1.2621**	**1.1919**	**1.2498**
11	**1.0411**	**1.0541**	**1.0550**	**1.0622**	**1.0717**	**1.0429**	**1.1381**	**1.3224**	**1.2064**	**1.1024**	**1.0411**	**1.0407**
12	**0.9759**	**0.9870**	**0.9879**	**0.9886**	**0.9913**	**0.9913**	**1.0055**	**1.0025**	**1.0022**	**0.9855**	**0.9762**	**0.9759**
13	**0.9981**	**1.0043**	**1.0042**	**0.9878**	**0.9722**	**0.9729**	**0.9861**	**0.9813**	**1.0010**	**1.0069**	**0.9988**	**0.9968**
14	**1.0359**	**1.1744**	**1.1230**	**1.1139**	**1.1368**	**0.9887**	**0.9953**	**0.9917**	**1.0069**	**1.0318**	**1.0297**	**1.0290**
15	**1.0776**	**1.0806**	**1.0301**	**1.0285**	**1.0453**	**1.0365**	**1.0383**	**1.0329**	**1.0375**	**1.0616**	**1.0678**	**1.0680**
16	**1.1480**	**1.1629**	**1.1105**	**1.1204**	**1.1404**	**1.1398**	**1.1497**	**1.1417**	**1.1352**	**1.1168**	**1.1343**	**1.1423**
17	**1.1920**	**1.2251**	**1.2303**	**1.2187**	**1.0856**	**1.1957**	**1.3729**	**1.6517**	**1.4872**	**1.2798**	**1.1686**	**1.1885**
18	**1.3415**	**1.3953**	**1.4093**	**1.3719**	**1.1463**	**1.4244**	**1.5595**	**1.7902**	**1.7022**	**1.4552**	**1.3326**	**1.3212**
19	**1.3659**	**1.4105**	**1.4011**	**1.4516**	**1.1545**	**1.4529**	**1.6050**	**1.8714**	**1.7432**	**1.5066**	**1.2772**	**1.3438**
20	**1.3894**	**1.4334**	**1.4240**	**1.4748**	**1.2245**	**1.3062**	**1.5219**	**1.9546**	**1.7169**	**1.4841**	**1.2628**	**1.3525**
21	**1.2859**	**1.3152**	**1.3177**	**1.3228**	**1.1738**	**1.1858**	**1.3915**	**1.7609**	**1.5828**	**1.3716**	**1.2407**	**1.2585**
22	**1.2120**	**1.2223**	**1.2224**	**1.2179**	**1.2045**	**1.2021**	**1.3679**	**1.6155**	**1.4592**	**1.3215**	**1.2318**	**1.1998**
23	**1.0702**	**1.0767**	**1.0788**	**1.0869**	**1.0750**	**1.0735**	**1.0837**	**1.0819**	**1.0939**	**1.0769**	**1.0691**	**1.0691**
24	**1.0486**	**1.0519**	**1.0534**	**1.0507**	**1.0280**	**1.0249**	**1.0310**	**1.0290**	**1.0469**	**1.0489**	**1.0440**	**1.0437**
25	**1.0258**	**1.0300**	**1.0316**	**1.0202**	**0.9954**	**0.9922**	**0.9993**	**0.9956**	**1.0103**	**1.0228**	**1.0195**	**1.0192**
26	**1.0709**	**1.2436**	**1.1861**	**1.1853**	**1.2134**	**1.0186**	**1.0115**	**1.0045**	**1.0261**	**1.0589**	**1.0608**	**1.0590**
27	**1.1314**	**1.1751**	**1.1204**	**1.1183**	**1.1453**	**1.0858**	**1.0722**	**1.0660**	**1.0739**	**1.1164**	**1.1143**	**1.1134**
28	**1.2477**	**1.2512**	**1.1939**	**1.2081**	**1.2228**	**1.2052**	**1.1918**	**1.1853**	**1.1929**	**1.1960**	**1.2223**	**1.2305**
29	**1.3413**	**1.3634**	**1.3680**	**1.3677**	**1.2133**	**1.4527**	**1.5736**	**1.7720**	**1.5298**	**1.3298**	**1.3015**	**1.3176**

30	1.4282	1.4522	1.4617	1.4642	1.0565	1.2828	1.3943	1.6184	1.6835	1.4441	1.4074	1.3798
31	1.6860	1.7403	1.7475	1.7547	1.0734	1.2637	1.3995	1.6708	1.7864	1.5648	1.4925	1.5978
32	1.8005	1.8601	1.8686	1.8639	1.2578	1.3339	1.4890	1.8994	1.8232	1.6102	1.4988	1.7178
33	1.4029	1.4408	1.4436	1.4597	1.3212	1.3322	1.5096	1.8692	1.6963	1.4770	1.3189	1.3681
34	1.3373	1.3531	1.3537	1.3650	1.3675	1.3578	1.4647	1.6770	1.5104	1.3929	1.3107	1.3160
35	1.0826	1.0843	1.0855	1.0966	1.0790	1.0749	1.1187	1.2892	1.2093	1.1146	1.0854	1.0779
36	1.0179	1.0194	1.0210	1.0364	1.0174	1.0097	1.0072	1.0068	1.0166	1.0091	1.0126	1.0131
37	1.0108	1.0151	1.0171	1.0338	1.0331	1.0240	1.0187	1.0174	1.0140	1.0046	1.0063	1.0070
38	1.0182	1.1742	1.1205	1.1309	1.1747	1.0059	0.9998	0.9993	1.0039	1.0156	1.0156	1.0153
39	1.1031	1.1686	1.1161	1.1280	1.1728	1.1080	1.0993	1.0970	1.0849	1.0965	1.0970	1.0974
40	1.4040	1.3916	1.3323	1.3581	1.4302	1.4698	1.4547	1.4398	1.4032	1.3799	1.3884	1.3943
41	1.8975	1.9304	1.9283	1.9375	1.5129	1.7996	1.9161	2.0870	1.9017	1.6845	1.8214	1.8410
42	1.9937	2.0489	2.0532	2.0591	1.3405	1.5049	1.6691	1.9928	1.9347	1.6979	1.8456	1.9131
43	2.2578	2.3145	2.2994	2.3769	1.1826	1.2955	1.4167	1.7577	1.8205	1.6248	1.7234	2.0790
44	1.9183	1.9686	1.9552	2.0226	1.1290	1.1592	1.3084	1.6909	1.6882	1.5070	1.4868	1.8028
45	1.5254	1.5681	1.5690	1.5951	1.3664	1.3324	1.5316	1.9316	1.7427	1.5262	1.4159	1.4917
46	1.4305	1.4576	1.4601	1.4931	1.4021	1.3760	1.5702	1.9258	1.7498	1.5288	1.3927	1.4215
47	1.2525	1.2695	1.2696	1.2794	1.3005	1.2797	1.3878	1.5328	1.4330	1.3225	1.2683	1.2567
48	1.1275	1.1414	1.1427	1.1278	1.1386	1.1486	1.1684	1.1641	1.1529	1.1488	1.1345	1.1327
49	1.1255	1.1384	1.1393	1.1156	1.1301	1.1338	1.1462	1.1384	1.1317	1.1459	1.1345	1.1309
50	1.2165	1.3890	1.3279	1.3258	1.4179	1.2646	1.2728	1.2641	1.2310	1.2358	1.2200	1.2185
51	1.2425	1.2701	1.2106	1.2255	1.3003	1.3274	1.3389	1.3261	1.2796	1.2422	1.2470	1.2554
52	1.4108	1.4129	1.3594	1.3702	1.4568	1.5267	1.5331	1.5071	1.4519	1.4065	1.4122	1.4239
53	1.6510	1.6917	1.7000	1.6739	1.3661	1.6056	1.6885	1.8991	1.7467	1.5371	1.6035	1.6283
54	1.9296	1.9827	1.9910	1.9807	1.1841	1.3865	1.4792	1.7302	1.8456	1.6221	1.8150	1.8783
55	2.0418	2.0824	2.0649	2.1376	1.2521	1.4315	1.5784	1.9331	1.8834	1.6635	1.6895	1.9618
56	1.6455	1.6827	1.6689	1.7285	1.2958	1.3840	1.5991	2.0519	1.8024	1.5702	1.4725	1.6028
57	1.4359	1.4774	1.4810	1.4925	1.3644	1.3930	1.6116	2.0057	1.8083	1.5667	1.4016	1.4162
58	1.3727	1.3927	1.3962	1.4042	1.4258	1.4135	1.4202	1.5897	1.4549	1.3947	1.3568	1.3620
59	1.0984	1.1079	1.1102	1.1079	1.1039	1.0947	1.0910	1.0853	1.0793	1.0986	1.0924	1.0909
60	1.0215	1.0289	1.0311	1.0194	1.0112	1.0045	1.0031	0.9965	1.0066	1.0286	1.0188	1.0165
61	1.0698	1.0769	1.0784	1.0718	1.0788	1.0753	1.0741	1.0690	1.0602	1.0751	1.0680	1.0664
62	1.2029	1.4016	1.3388	1.3574	1.4746	1.2851	1.2714	1.2673	1.2116	1.2136	1.2049	1.2038
63	1.2938	1.3207	1.2591	1.2765	1.3642	1.3644	1.3397	1.3370	1.2725	1.2960	1.2935	1.2915
64	1.4493	1.3835	1.3187	1.3605	1.4623	1.5829	1.5447	1.5409	1.4425	1.4287	1.4463	1.4524
65	1.8187	1.8378	1.8364	1.8513	1.5348	1.6835	1.8128	2.1547	1.9214	1.6963	1.7452	1.7653
66	1.8654	1.9123	1.9208	1.9220	1.2138	1.3924	1.4919	1.7687	1.8118	1.5912	1.7562	1.7934
67	2.2309	2.2936	2.2795	2.3570	1.1303	1.2842	1.4109	1.7295	1.8490	1.6399	1.7690	2.0736
68	1.9754	2.0131	1.9958	2.0662	1.1684	1.2144	1.3826	1.7928	1.8026	1.5942	1.6048	1.8980
69	1.7245	1.7767	1.7783	1.8012	1.4971	1.4525	1.6367	2.0508	1.8413	1.6320	1.5844	1.7190
70	1.5060	1.5383	1.5420	1.5684	1.4961	1.4915	1.6850	2.0395	1.8434	1.6157	1.4853	1.5069
71	1.1912	1.2098	1.2135	1.2268	1.2440	1.2490	1.3792	1.5776	1.4281	1.2874	1.2135	1.1969
72	1.1733	1.1892	1.1936	1.1963	1.2131	1.2254	1.2386	1.2286	1.2096	1.1821	1.1806	1.1843
Mean	1.3552	1.3916	1.3774	1.3897	1.2052	1.2403	1.3196	1.4841	1.4165	1.3095	1.2848	1.3307

For the final month 72, the lowest index value in Table A26 is 1.1733 and the highest is 1.2386. From Table A25, the lowest index value for month 72 was 1.0793 and the highest was 1.2241. Thus using maximum overlap bilateral Fisher indexes in place of carry forward Fisher indexes has led to a narrower spread of final index values when different base periods are used. The arithmetic mean of the month 72 means recorded in the last row of Table A25 is 1.1589 and the corresponding arithmetic mean of the month 72 means recorded in the last row of Table A 26 is 1.1856. Thus for these alternative fixed base Fisher indexes, the use of carry forward prices led to final index values which on average are 2.68 percentage points below the corresponding maximum overlap indexes. This is a significant downward bias over the 6 year period.

The average index value over all observations listed in Table A25 is 1.2065. The corresponding average index value over all observations listed in Table A26 is 1.3145, an increase of 10.8 percentage points. The use of maximum overlap indexes in place of their carry forward counterparts has greatly increased seasonal fluctuations and led to fixed base Fisher indexes which have much larger seasonal peaks. Thus the use of carry forward prices has led to Fisher fixed base indexes which have a lower trend and much lower seasonal fluctuations than their maximum overlap counterpart indexes.

References

Allen, R.C. and W.E. Diewert (1981), "Direct versus Implicit Superlative Index Number Formulae", *Review of Economics and Statistics* 63, 430-435.

Alterman, W.F., W.E. Diewert and R.C. Feenstra, (1999), *International Trade Price Indexes and Seasonal Commodities*, Bureau of Labor Statistics, Washington D.C.

Anderson, O. (1927), "On the Logic of the Decomposition of Statistical Series into Separate Components", *Journal of the Royal Statistical Society* 90, 548-569.

Armknecht, P.A. and F. Maitland-Smith (1999), "Price Imputation and Other Techniques for Dealing with Missing Observations, Seasonality and Quality Change in Price Indices," IMF Working Paper No. 99/78, Washington, D.C., June.

Aten, B. and A. Heston (2009), "Chaining Methods for International Real Product and Purchasing Power Comparisons: Issues and Alternatives", pp. 245-273 in *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, D.S. Prasada Rao (ed.), Cheltenham UK: Edward Elgar.

Australian Bureau of Statistics (2016), "Making Greater Use of Transactions Data to Compile the Consumer Price Index", Information Paper 6401.0.60.003, November 29, Canberra: ABS.

Baldwin, A. (1990), "Seasonal Baskets in Consumer Price Indexes", *Journal of Official Statistics* 6:3, 251-273.

Balk, B.M. (1980a), *Seasonal Products in Agriculture and Horticulture and Methods for Computing Price Indices*, Statistical Studies no. 24, The Hague: Netherlands Central Bureau of Statistics.

Balk, B.M. (1980b), "Seasonal Commodities and the Construction of Annual and Monthly Price Indexes", *Statistische Hefte* 21:2, 110-116.

Balk, B.M. (1980c), "A Method for Constructing Price Indices for Seasonal Commodities", *Journal of the Royal Statistical Society A* 143, 68-75.

Balk, B.M. (1981), "A Simple Method for Constructing Price Indices for Seasonal Commodities", *Statistische Hefte* 22 (1), 1–8.

Balk, B.M. (1996), "A Comparison of Ten Methods for Multilateral International Price and Volume Comparisons", *Journal of Official Statistics* 12, 199-222.

Balk, B.M. (2008), *Price and Quantity Index Numbers*, New York: Cambridge University Press.

Bean, L. H. and O. C. Stine (1924), "Four Types of Index Numbers of Farm Prices", *Journal of the American Statistical Association* 19, 30-35.

Carli, Gian-Rinaldo, (1804), "Del valore e della proporzione de' metalli monetati", pp. 297-366 in *Scrittori classici italiani di economia politica*, Volume 13, Milano: G.G. Destefanis (originally published in 1764).

Caves, D.W., L.R. Christensen and W.E. Diewert (1982), "Multilateral Comparisons of Output, Input and Productivity using Superlative Index Numbers", *Economic Journal* 92, 73-86.

Court, A.T. (1939), "Hedonic Price Indexes with Automotive Examples", pp. 99-117 in The *Dynamics of Automobile Demand*, New York: General Motors Corporation.

Crump, N. (1924), "The Interrelation and Distribution of Prices and their Incidence Upon Price Stabilization", *Journal of the Royal Statistical Society* 87, 167-206.

Dalén, J. (1992), "Computing Elementary Aggregates in the Swedish Consumer Price Index", *Journal of Official Statistics* 8, 129-147.

de Haan, J. (2015), "Rolling Year Time Dummy Indexes and the Choice of Splicing Method", Room Document at the 14th meeting of the Ottawa Group, May 22, Tokyo. http://www.stat.go.jp/english/info/meetings/og2015/pdf/t1s3room

de Haan, J. and H. van der Grient (2011), "Eliminating Chain Drift in Price Indexes Based on Scanner Data", *Journal of Econometrics* 161, 36-46.

Diewert, W.E. (1976), "Exact and Superlative Index Numbers", *Journal of Econometrics* 4, 114-145.

Diewert, W.E. (1978), "Superlative Index Numbers and Consistency in Aggregation", *Econometrica* 46, 883-900.

Diewert, W.E. (1983), "The Treatment of Seasonality in a Cost of Living Index", pp. 1019-1045 in *Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada.

Diewert, W.E. (1988), "Test Approaches to International Comparisons", pp. 67-86 in *Measurement in Economics: Theory and Applications of Economic Indices*, W. Eichhorn (ed.), Heidelberg: Physica-Verlag.

Diewert, W.E. (1992), "Fisher Ideal Output, Input and Productivity Indexes Revisited", *Journal of Productivity Analysis* 3, 211-248.

Diewert, W.E. (1995), "Axiomatic and Economic Approaches to Elementary Price Indexes", Discussion Paper No. 95-01, Department of Economics, University of British Columbia, Vancouver, Canada.

Diewert, W.E. (1998), "High Inflation, Seasonal Commodities and Annual Index Numbers", *Macroeconomic Dynamics* 2, 456-471.

Diewert, W.E. (1999a), "Index Number Approaches to Seasonal Adjustment", *Macroeconomic Dynamics* 3, 1-21.

Diewert, W.E. (1999b), "Axiomatic and Economic Approaches to International Comparisons", pp. 13-87 in *International and Interarea Comparisons of Income, Output and Prices*, A. Heston and R.E. Lipsey (eds.), Studies in Income and Wealth, Volume 61, Chicago: The University of Chicago Press.

Diewert, W.E. (2009), "Similarity Indexes and Criteria for Spatial Linking", pp. 183-216 in *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, D.S. Prasada Rao (ed.), Cheltenham, UK: Edward Elgar.

Diewert, W.E. (2021a), "Elementary Indexes", Draft Chapter 6 in *Consumer Price Index Theory*, Washington D.C.: International Monetary Fund, published online at: https://www.imf.org/en/Data/Statistics/cpi-manual.

Diewert, W.E. (2021b), "The Chain Drift Problem and Multilateral Indexes", Draft Chapter 7 in *Consumer Price Index Theory*, Washington D.C.: International Monetary Fund, published online at:
https://www.imf.org/en/Data/Statistics/cpi-manual.

Diewert, W.E. (2021c), "Quality Adjustment Methods", Draft Chapter 8 in *Consumer Price Index Theory*, Washington D.C.: International Monetary Fund, published online at: https://www.imf.org/en/Data/Statistics/cpi-manual.

Diewert, W.E., W.F. Alterman and R.C. Feenstra (2012), "Time Series versus Index Number Methods of Seasonal Adjustment", pp. 29-52 in *Price and Productivity Measurement: Volume 2—Seasonality*, Chapter 3 in Diewert, W.E., B.M. Balk, D. Fixler, K.J. Fox and A.O. Nakamura, Trafford Press.

Diewert, W.E., Y. Finkel and Y. Artsev (2011), "Empirical Evidence on the Treatment of Seasonal Products: The Israeli CPI Experience", pp. 53-78 in *Price and Productivity Measurement: Volume 2—Seasonality*, Chapter 4 in Diewert, W.E., B.M. Balk, D. Fixler, K.J. Fox and A.O. Nakamura, Trafford Press.

Diewert, W.E. and K.J. Fox (2017), "Output Growth and Inflation Across Space and Time", *EURONA* 2017:1, 7-40.

Diewert, W.E. and K.J. Fox (2018), "Addendum to Output Growth and Inflation Across Space and Time", *EURONA* 2018:2, 1-10.

Diewert, W.E. and K.J. Fox (2020) "Substitution Bias in Multilateral Methods for CPI Construction Using Scanner Data," *Journal of Business and Economic Statistics*, published online at: https://doi.org/10.1080/07350015.2020.1816176

Dutot, Charles, (1738), *Réflexions politiques sur les finances et le commerce*, Volume 1, La Haye: Les frères Vaillant et N. Prevost.

Eichhorn, W. (1978), *Functional Equations in Economics*, Reading, MA: Addison-Wesley Publishing Company.

Eltetö, Ö., and Köves, P. (1964), "On a Problem of Index Number Computation Relating to International Comparisons", (in Hungarian), *Statisztikai Szemle* 42, 507-518.

Fisher, I. (1922), *The Making of Index Numbers*, Boston: Houghton Mifflin Co.

Flux, A.W. (1921), "The Measurement of Price Change", *Journal of the Royal Statistical Society* 84, 167-199.

Gini, C. (1924), "Quelques considérations au sujet de la construction des nombres indices des prix et des questions analogues", *Metron* 4:1, 3-162.

Gini, C. (1931), "On the Circular Test of Index Numbers", *Metron* 9:9, 3-24.

Hill, R.J. (1997), "A Taxonomy of Multilateral Methods for Making International Comparisons of Prices and Quantities", *Review of Income and Wealth* 43(1), 49-69.

Hill, R.J. (1999a), "Comparing Price Levels across Countries Using Minimum Spanning Trees", *The Review of Economics and Statistics* 81, 135-142.

Hill, R.J. (1999b), "International Comparisons using Spanning Trees", pp. 109-120 in *International and Interarea Comparisons of Income, Output and Prices*, A. Heston and R.E. Lipsey (eds.), Studies in Income and Wealth Volume 61, NBER, Chicago: The University of Chicago Press.

Hill, R.J. (2001), "Measuring Inflation and Growth Using Spanning Trees", *International Economic Review* 42, 167-185.

Hill, R.J. (2004), "Constructing Price Indexes Across Space and Time: The Case of the European Union", *American Economic Review* 94, 1379-1410.

Hill, R.J. (2009), "Comparing Per Capita Income Levels Across Countries Using Spanning Trees: Robustness, Prior Restrictions, Hybrids and Hierarchies", pp. 217-244 in *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, D.S. Prasada Rao (ed.), Cheltenham UK: Edward Elgar.

Hill, R.J. and M.P. Timmer (2006), "Standard Errors as Weights in Multilateral Price Indexes", *Journal of Business and Economic Statistics* 24:3, 366-377.

Inklaar, R. and W.E. Diewert (2016), "Measuring Industry Productivity and Cross-Country Convergence", *Journal of Econometrics* 191, 426-433.

Ivancic, L., W.E. Diewert and K.J. Fox (2009), "Scanner Data, Time Aggregation and the Construction of Price Indexes", Discussion Paper 09-09, Department of Economics, University of British Columbia, Vancouver, Canada.

Ivancic, L., W.E. Diewert and K.J. Fox (2011), "Scanner Data, Time Aggregation and the Construction of Price Indexes", *Journal of Econometrics* 161, 24-35.

Jevons, W.S., (1865), "The Variation of Prices and the Value of the Currency since 1782", *Journal of the Statistical Society of London* 28, 294-320; reprinted in *Investigations in Currency and Finance* (1884), London: Macmillan and Co., 119-150.

Keynes, J.M. (1909), "The Method of Index Numbers with Special Reference to the Measurement of General Exchange Value", reprinted as pp. 49-156 in *The Collected Writings of John Maynard Keynes* (1983), Volume 11, D. Moggridge (ed.), Cambridge: Cambridge University Press.

Keynes, J.M. (1930), *Treatise on Money*, Volume 1, London: Macmillan.

Krsinich, F. (2016), "The FEWS Index: Fixed Effects with a Window Splice', *Journal of Official Statistics* 32, 375-404.

Laspeyres, E. (1871), "Die Berechnung einer mittleren Waarenpreissteigerung", *Jahrbücher für Nationalökonomie und Statistik* 16, 296-314.

Lowe, J. (1823), *The Present State of England in Regard to Agriculture, Trade and Finance*, second edition, London: Longman, Hurst, Rees, Orme and Brown.

Mendershausen, H. (1937), "Annual Survey of Statistical Technique: Methods of Computing and Eliminating Changing Seasonal Fluctuations", *Econometrica* 5, 234-262.

Mitchell, W.C. (1927), *Business Cycles*, New York: National Bureau of Economic Research.

Mudgett, B.D. (1955), "The Measurement of Seasonal Movements in Price and Quantity Indexes", *Journal of the American Statistical Association* 50, 93-98.

O' Donnell, G. and C. Yélou (2021), *Adjusted Price Index and Monthly Adjusted Consumer Expenditure Basket Weights*, Catalogue no. 62F0014M, Release date: November 10, 2021, Ottawa: Statistics Canada.

Paasche, H. (1874), "Über die Preisentwicklung der letzten Jahre nach den Hamburger Borsennotirungen", *Jahrbücher für Nationalökonomie und Statistik* 12, 168-178.

Stone, R. (1956), *Quantity and Price Indexes in National Accounts*, Paris: OECD.

Summers, R. (1973), "International Comparisons with Incomplete Data", *Review of Income and Wealth* 29:1, 1-16.

Szulc, B.J. (1964), "Indices for Multiregional Comparisons", (in Polish), *Przeglad Statystyczny* 3, 239-254.

Szulc, B.J. (1983), "Linking Price Index Numbers," pp. 537-566 *in Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada.

Szulc, B.J. (1987), “Price Indices below the Basic Aggregation Level”, *Bulletin of Labour Statistics* 2, 9-16.

Theil, H. (1967), *Economics and Information Theory*, Amsterdam: North-Holland Publishing.

Törnqvist, L. (1936), “The Bank of Finland's Consumption Price Index”, *Bank of Finland Monthly Bulletin* 10, 1-8.

Törnqvist, L. and E. Törnqvist (1937), Vilket är förhällandet mellan finska markens och svenska kronans köpkraft?”, *Ekonomiska Samfundets Tidskrift* 39, 1-39 reprinted as pp. 121-160 in *Collected Scientific Papers of Leo Törnqvist*, Helsinki: The Research Institute of the Finnish Economy, 1981.

Triplett, J. E. and R. J. McDonald (1977), “Assessing the Quality Error in Output Measures: The Case of Refrigerators”, *The Review of Income and Wealth* 23:2, 137-156.

Turvey, R. (1979), “The Treatment of Seasonal Items in Consumer Price Indices”, *Bulletin of Labour Statistics*, Fourth Quarter, International Labour Office, Geneva, 13-33.

Walsh, C.M. (1901), *The Measurement of General Exchange Value*, New York: Macmillan and Co.

Walsh, C.M. (1921), “Discussion”, *Journal of the American Statistical Association* 17, 537-544.

Young, A. (1812), *An Inquiry into the Progressive Value of Money in England as Marked by the Price of Agricultural Products*, London.

Yule, G.U. (1921), “Discussion of Mr. Flux’s Paper”, *Journal of the Royal Statistical Society* 84, 199-202.

Zarnowitz, V. (1961), “Index Numbers and the Seasonality of Quantities and Prices”, pp. 233-304 in *The Price Statistics of the Federal Government*, G.J. Stigler (Chairman), New York: National Bureau of Economic Research.

Zhang, L.-C., I. Johansen and R. Nygaard (2019), “Tests for Price Indices in a Dynamic Item Universe*”, Journal of Official Statistics*, 35:3, 683–697.